

YI'S UNIQUE RANGE SET CONSTRUCTION IN THE NUMBER FIELD CASE

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ABSTRACT. H. X. Yi's construction of unique range sets for entire functions is translated to the number theory setting to illustrate that his construction would work in the number theory setting if one knew a version of Schmidt's Subspace Theorem with truncated counting functions.

1. INTRODUCTION

A finite set \mathcal{S} of the complex numbers is called a *unique range set* (counting multiplicity) for entire functions if whenever $f^*\mathcal{S} = g^*\mathcal{S}$ (pull-back of \mathcal{S} as a divisor) for two non-constant entire functions f and g , then one must have $f = g$. It is easy to see that a unique range set for entire functions must contain at least five points. Indeed, let $\mathcal{S} = \{a_1, b_1, a_2, b_2\}$ be a four point set, let L be the Möbius involution such that $L(a_j) = b_j$ and $L(b_j) = a_j$. Then, if f is an entire function omitting $L(\infty)$, then $L \circ f$ is also an entire function and $(L \circ f)^*\mathcal{S} = f^*(L^*\mathcal{S}) = f^*\mathcal{S}$. In [5], Yi constructed examples of unique range sets for entire functions of cardinality ≥ 7 . It seems to be a difficult open problem to determine if there can be a unique range set for entire functions with five or six elements.

The main tool in Yi's construction is Nevanlinna theory. As is now standard, one can try to transpose Yi's result to number theory. All constructions of unique range sets known to me make use of the Second Main Theorem with truncated counting functions. Thus, no existing construction of unique range sets for entire functions will give an analogous theorem in number theory, except in the case of number fields with finite unit group. Some constructions also make more complicated use of differentiation, which poses an addition challenge in adapting them to the number theory setting. However, Yi's construction only makes use of the truncated second main theorem, and thus one can translate his construction to number theory, assuming a conjecture that Schmidt's Subspace Theorem remains true with appropriately truncated counting functions.

The purpose of this note is to translate Yi's construction to the number theory setting and to highlight where truncated counting functions are used. This note contains no new ideas; it is simply a translation of Yi's paper into number theory.

I begin by defining the number theory analogs. Let k be a number field, let S be a finite set of places of k , let \mathcal{O}_S denote the ring of S -integers in k , and let h denote an additive height function on k . Let \mathcal{S} be a finite set of S -units in k . I will call a

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sequence x_j of S -integers in k *admissible* if $h(x_j) \rightarrow \infty$. Two admissible sequences x_j and y_j of S -integers are said to *share* \mathcal{S} (counting multiplicity) if for all places v of k not in S , we have

$$\left| \prod_{s \in \mathcal{S}} x_j - s \right|_v = \left| \prod_{s \in \mathcal{S}} y_j - s \right|_v.$$

The set \mathcal{S} is called a *unique range set* (counting multiplicity) for \mathcal{O}_S if whenever x_j and y_j are admissible sequences of S -integers that share \mathcal{S} , then $x_j = y_j$ for all but finitely many j .

From the definition of sharing \mathcal{S} , one sees immediately that it is useful to consider the polynomial

$$P_{\mathcal{S}}(X) = \prod_{s \in \mathcal{S}} (X - s).$$

If x_j and y_j are admissible sequences sharing \mathcal{S} , then this precisely means that

$$(1) \quad P_{\mathcal{S}}(x_j) = u_j P_{\mathcal{S}}(y_j)$$

for a sequence of S -units u_j . This leads to the notion of strong uniqueness polynomials. A polynomial P is called a *strong uniqueness polynomial* for \mathcal{O}_S if whenever one has two admissible sequences x_j and y_j of S -integers such that $P(x_j) = cP(y_j)$ for some S -unit c , then one must have $x_j = y_j$ for all but finitely many j . By Faltings's theorem, one sees that a polynomial P is a strong uniqueness polynomial for the rings of S -integers in all number fields if and only if the 2-variable polynomials $P(X) - cP(Y)$ for $c \neq 0$ do not have any linear or quadratic factors (over $\mathbf{Q}^a[X, Y]$), except for the linear factor $X - Y$ when $c = 1$. See [1], [2], and [3] for various criteria that can therefore be used to give concrete examples of uniqueness polynomials for \mathcal{O}_S . Let me also remark here that if the group of units in \mathcal{O}_S is a finite group, then there is no difference between the concept of unique range set and strong uniqueness polynomial because there are only finitely many possibilities of u_j in equation (1).

As in [4], for x an element of k , we define the *counting function* (of zeros) by

$$N(x) = \frac{1}{[k : \mathbf{Q}]} \sum_{v \notin S} \max\{0, \text{ord}_v(x)\} [k_v : \mathbf{Q}_v] \log p_v,$$

where k_v denotes the completion of k at the place v and p_v is the prime in \mathbf{Q} which v lies above. Similarly, the *counting function truncated to multiplicity ℓ* , where ℓ is a positive integer, is defined by

$$N^{(\ell)}(x) = \frac{1}{[k : \mathbf{Q}]} \sum_{v \notin S} \min\{\max\{0, \text{ord}_v(x)\}, \ell\} [k_v : \mathbf{Q}_v] \log p_v,$$

For Yi's construction to work, one must assume the following conjectural strengthening of Schmidt's subspace theorem.

Conjecture 1. *Let L_1, \dots, L_q be linear forms in $r + 1$ -variables with coefficients in k determining q hyperplanes in general position in \mathbf{P}^n . Let x_j^0, \dots, x_j^r be $r + 1$ sequences of S -integers, at least one of which is admissible, such that for each j and each place v not in S ,*

$$\max\{|x_j^0|_v, \dots, |x_j^r|_v\} = 1,$$

and such that there is no linear form L such that $L(x_j^0, \dots, x_j^r) = 0$ for infinitely many j . Let $\varepsilon > 0$. Then, for all j sufficiently large,

$$(q - r - 1 - \varepsilon) \max\{h(x_j^0), \dots, h(x_j^r)\} \leq \sum_{i=1}^q N^{(r)}(L_i(x_j^0, \dots, x_j^r)).$$

Remark. If the counting functions on the right were not truncated, this would be Schmidt's Subspace Theorem. We will only need the conjecture when $r \leq 2$.

Corollary 2. *Assuming Conjecture 1 when $r = 1$, if x_j and y_j are sequences of S -integers with x_j admissible, and if A , B , and C are non-zero constants such that*

$$Ax_j + By_j = C$$

for all but finitely many j , then

$$(1 - \varepsilon)h(x_j) \leq N^{(1)}(x_j) + N^{(1)}(y_j)$$

for all but finitely many j .

Proof. Apply the conjecture with $r = 1$ with $x_j^0 = 1$, with $x_j^1 = x_j$, and with the three linear forms:

$$L_1(x^0, x^1) = x^0, \quad L_2(x^0, x^1) = x^1, \quad \text{and} \quad L_3(x^0, x^1) = Cx^0 - Ax^1. \quad \square$$

2. YI'S CONSTRUCTION

Theorem 3 (H. X. Yi [5, Theorem 1]). *Assume Conjecture 1 holds when $r \leq 2$. Let n and m be relatively prime positive integers such that $n > 2m + 4$. Let a and b be S -units such that the polynomial $P(X) = X^n + aX^{n-m} + b$ has no multiple roots and that the roots of P are S -units. Then, the set of zeros of P is a unique range set for \mathcal{O}_S .*

Example. *Let $P(X) = X^7 + X^6 + 1$. Then, assuming Conjecture 1, the zeros of P form a unique range set for \mathcal{O}_S for any number field k containing the roots of P and for any finite set of places (containing all the Archimedean places) S large enough that all the roots of P are S -integers.*

Proof of Theorem 3. We adopt the convention that throughout the proof all height inequalities hold for all but finitely many terms and ε is a positive number that is adjusted as necessary.

Let $\mathcal{S} = \{s_1, \dots, s_n\}$ be the zeros of P . Assume that x_j and y_j are two admissible sequences of S -integers that share \mathcal{S} , and so there are S -units u_j such that $P(x_j) = u_j P(y_j)$.

By Roth's Theorem, the Product formula, and the assumption that x_j and y_j share \mathcal{S} ,

$$(n - 1 - \varepsilon)h(y_j) \leq \sum_{s \in \mathcal{S}} N(y_j - s) = \sum_{s \in \mathcal{S}} N(x_j - s) \leq nh(x_j) + O(1).$$

Thus by symmetry, we have $h(x_j) = O(h(y_j))$ and $h(y_j) = O(h(x_j))$. The comparability in height is an important feature of sequences sharing finite sets. Because $u_j = P(x_j)/P(y_j)$, we have by elementary properties of heights,

$$h(u_j) \leq h(P(x_j)) + h(P(y_j)) = O(h(x_j)).$$

Consider the following auxiliary sequences

$$\begin{aligned}\eta_j &= -\frac{1}{b}x_j^{n-m}(x_j^m + a) \\ \zeta_j &= \frac{1}{b}y_j^{n-m}(y_j^m + a)u_j\end{aligned}$$

Because n is somewhat larger than m and η_j and ζ_j begin with something to the $n - m$ power, they will have places dividing them with moderate multiplicity. Exploiting this extra multiplicity is where we will use Conjecture 1, and finding a way to exploit this sort of multiplicity without referring to an unproven conjecture is the main obstacle in using existing analytic constructions of unique range sets in the number field setting.

Now, notice that

$$\eta_j + u_j + \zeta_j = \frac{-1}{b}(P(x_j) - b) + u_j + \frac{1}{b}(P(x_j) - bu_j) = 1.$$

Assume for the moment that there is no linear form in three variables L such that $L(\eta_j, u_j, \zeta_j) = 0$ for infinitely many j . Then, we may apply Conjecture 1 to conclude that

$$(1 - \varepsilon) \max\{h(\eta_j), h(u_j), h(\zeta_j)\} \leq N^{(2)}(\eta_j) + N^{(2)}(u_j) + N^{(2)}(\zeta_j) + N^{(2)}(\eta_j + u_j + \zeta_j).$$

Because u_j is an S -unit and $\eta_j + u_j + \zeta_j = 1$, $N^{(2)}(u_j) = N^{(2)}(\eta_j + u_j + \zeta_j) = 0$. Clearly,

$$N^{(2)}(\eta_j) \leq 2N^{(1)}(x_j) + N(x_j^m + a) \quad \text{and} \quad N^{(2)}(\zeta_j) \leq 2N^{(1)}(y_j) + N(y_j^m + a).$$

Because the counting functions are bounded by the heights (Product Formula) and $h(x_j^m + a)$ is comparable to $mh(x_j)$ and similarly for y_j , we get

$$(1 - \varepsilon) \max\{h(\eta_j), h(u_j), h(\zeta_j)\} \leq (2 + m)h(x_j) + (2 + m)h(y_j) + O(1).$$

Also,

$$h(\eta_j) \geq nh(x_j) + O(1),$$

so

$$(n - \varepsilon)h(x_j) \leq (2 + m)h(x_j) + (2 + m)h(y_j) + O(1).$$

Reversing the roles of x_j and y_j we also get

$$(n - \varepsilon)h(y_j) \leq (2 + m)h(x_j) + (2 + m)h(y_j) + O(1).$$

Adding the previous two inequalities give

$$(n - \varepsilon)(h(x_j) + h(y_j)) \leq (4 + 2m)(h(x_j) + h(y_j)) + O(1),$$

which contradicts the assumption that $n > 2m + 4$.

The proof is completed by the following proposition.

Proposition 4. *Let c_1, c_2 and c_3 be in k and not all zero. With the notation as in the theorem, if $x_j \neq y_j$ for infinitely many j , then*

$$c_1\eta_j + c_2u_j + c_3\zeta_j = 0$$

for at most finitely many j .

Proof. Suppose $c_1\eta_j + c_2u_j + c_3\zeta_j = 0$ for infinitely many j . Clearly at least two of the c_i are non-zero.

Case $c_1 = 0$: Then, $\zeta_j = (c_2/c_3)u_j$ and so $y_j^{n-m}(y_j^m + a) = bc_2/c_3$, which contradicts the assumption that $h(y_j) \rightarrow \infty$. So this case does not occur.

We may now assume $c_1 \neq 0$. Because $\eta_j + u_j + \zeta_j = 1$, we can remove η_j to get $C_2u_j + C_3\zeta_j = 1$, where

$$C_i = 1 - \frac{c_i}{c_1}.$$

Case $C_2 \neq 0$ and $C_3 \neq 0$: In this case,

$$C_3\zeta_j u_j^{-1} - u_j^{-1} = -C_2,$$

and so we can apply Corollary 2 to conclude

$$(1 - \varepsilon)h(\zeta_j u_j^{-1}) \leq N^{(1)}(\zeta_j u_j^{-1})$$

for infinitely many j . But,

$$\zeta_j u_j^{-1} = \frac{1}{b}y_j^{n-m}(y_j^m + a),$$

and hence $h(\zeta_j u_j^{-1}) = nh(y_j) + O(1)$. Also,

$$N^{(1)}(\zeta_j u_j^{-1}) \leq N^{(1)}(y_j) + N^{(1)}(y_j^m + a) \leq (m+1)h(y_j) + O(1).$$

This contradicts $n > 2m + 4$.

Case $C_2 = 0$: In this case,

$$y_j^{n-m}(y_j^m + a) = bC_3^{-1}u_j^{-1}.$$

Enlarging S if necessary, we may assume bC_3^{-1} is a unit in \mathcal{O}_S . This implies that y_j and $y_j^m + a$ are S -units for all j , which is a contradiction to the S -integer version of Picard's theorem.

Case $C_3 = 0$: In this case we see that $u_j = C_2^{-1}$ for infinitely many j . Thus, either $x_j = y_j$ for all but finitely many of these j , or P is not a strong uniqueness polynomial. We have already remarked that if P is a strong uniqueness polynomial for entire functions, then it is also a strong uniqueness polynomial for S -integers. Thus, the proof is completed by showing P is a strong uniqueness polynomial as in Yi [5], or alternatively as in [3]. This is where the assumption that n and m are relatively prime is used. \square

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