The Potential Theoretic Method of Eremenko & Sodin Part III: Eremenko & Sodin's original paper: their proof of Shiffman's Conjecture

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- A. Eremenko and M. Sodin, Distribution of values of meromorphic functions and meromorphic curves from the standpoint of potential theory. (Russian), *Algebra i Analiz* **3** (1991), 131–164.
- English translation in: *St. Petersburg Math. J.* **3** (1992), 109–136.

Shiffman's Conjecture

Theorem (Eremenko & Sodin)

Let $Q_1, \ldots, Q_q$ be q homogeneous forms of degree d_j whose corresponding hypersurfaces are in general position in $\mathbf{P}^n$, and let $f : \mathbf{C} \to \mathbf{P}^n$ be a holomorphic curve such that none of the $Q_j \circ f$ are identically zero. Then,

$$\text{[unintegrated form]} \quad (q-2n)A_f(r) \leq \sum_{j=1}^{q} d_j^{-1} n_{Q_j \circ f}(r) + o(A_f(r))$$

$$\text{[integrated form]} \quad (q-2n)T_f(r) \leq \sum_{j=1}^{q} d_j^{-1} N_{Q_j \circ f}(r) + o(T_f(r)),$$

where both inequalities hold as $r \to \infty$ outside exceptional sets of finite logarithmic measure.

Theorem (Eremenko & Sodin)

Let $Q_1, \ldots, Q_q$ be q homogeneous forms of degree d_j whose corresponding hypersurfaces are in general position in $\mathbf{P}^n$, and let $f : \mathbf{C} \to \mathbf{P}^n$ be a holomorphic curve such that none of the $Q_j \circ f$ are identically zero. Then,

$$(q - 2n) T_f(r) \quad \leq \quad \sum_{j=1}^{q} d_j^{-1} N_{Q_j \circ f}(r) + o(T_f(r)),$$

as $r \to \infty$ outside an exceptional set of finite logarithmic measure.

Remark

The authors also prove the integrated inequality in the case of slowly moving hypersurfaces.

Theorem (Eremenko & Sodin)

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as $r \to \infty$ outside an exceptional set of finite logarithmic measure.

Remark

Compare with Ru's version using the method of Evertse and Ferretti:

$$(q - n - 1 - \varepsilon) T_f(r) \leq \sum_{j=1}^{q} d_j^{-1} N_{Q_j \circ f}(r).$$

under the additonal assumption that f is algebraically non-degenerate.

Theorem (Eremenko & Sodin)

Let $Q_1, \ldots, Q_q$ be q homogeneous forms of degree d_j whose corresponding hypersurfaces are in general position in $\mathbf{P}^n$, and let $f : \mathbf{C} \to \mathbf{P}^n$ be a holomorphic curve such that none of the $Q_j \circ f$ are identically zero. Then,

$$(q-2n)T_f(r) \;\leq\; \sum_{j=1}^{q} d_j^{-1} N_{Q_j \circ f}(r) + o(T_f(r)),$$

as $r \to \infty$ outside an exceptional set of finite logarithmic measure.

Remark

To date, there is no proof for a number-theoretic analog of Eremenko & Sodin's inequality.

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as $r \to \infty$ outside an exceptional set of finite logarithmic measure.

Remark

Although not stated in their paper, for reasons explained yesterday, the method works just as well for $f : \mathbf{C} \to M \subset \mathbf{P}^N$, where M is a closed subset and the hypersurfaces Q_j are n-general with respect to M.

Theorem (Eremenko & Sodin)

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as $r \to \infty$ outside an exceptional set of finite logarithmic measure.

Remark

Their method does not easily generalize if the domain $\mathbf{C}$ is replaced by a more general manifold. They make heavy use of planar geometry in their proofs, also beyond the Rickman covering lemma.

Theorem (Eremenko & Sodin)

Let $Q_1, \ldots, Q_q$ be q homogeneous forms of degree d_j whose corresponding hypersurfaces are in general position in $\mathbf{P}^n$, and let $f : \mathbf{C} \to \mathbf{P}^n$ be a holomorphic curve such that none of the $Q_j \circ f$ are identically zero. Then,

$$(q-2n)T_f(r) \quad \leq \quad \sum_{j=1}^{q} d_j^{-1} N_{Q_j \circ f}(r) + o(T_f(r)),$$

as $r \to \infty$ outside an exceptional set of finite logarithmic measure.

Remark

Note that there is no "ramification" or "trunctation" in their inequality. This method cannot prove something like

$$(q-2n)T_f(r) \leq \sum_{j=1}^{q} N^{(1)}_{Q_j \circ f}(r) + o(T_f(r)).$$

Theorem (Eremenko & Sodin)

Let $Q_1, \ldots, Q_q$ be q homogeneous forms of degree d_j whose corresponding hypersurfaces are in general position in $\mathbf{P}^n$, and let $f : \mathbf{C} \to \mathbf{P}^n$ be a holomorphic curve such that none of the $Q_j \circ f$ are identically zero. Then,

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as $r \to \infty$ outside an exceptional set of finite logarithmic measure.

Remark

Their method should be effective, in principle. For those interested in the precise structure of error terms, it could be interesting to work out the precise error term given by their method, even in the case $n = 1$ and compare with the error terms given by logarithmic derivatives and negative curvature.

Some remarks about the proof

- As yesterday, one sets
$$u = \log\left(|f_0|^2 + \cdots + |f_n|^2\right),$$
which is subharmonic and non-harmonic if f is non-constant.

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- Like yesterday,
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- Like yesterday,
$$\max_{j \in I} u_j = u + O(1)$$
for any index set I of cardinality $n+1$.
- For the moving targets case, one gets
$$\max_{j \in I} u_j = u + o(T_f(r)).$$

The potential theoretic result

As yesterday, one applies the Rickman covering lemma and rescales to get functions v and v_j on $D2)$, the disc of radius 2. As before v is sub-harmonic and non-harmonic. However, this time the v_j are NOT harmonic. They are what is known as δ-subharmonic, meaning they can be written as the difference of two sub-harmonic functions.

Theorem (Eremenko & Sodin)

Let $L, M > 0$ and $q, n \in \mathbf{N}$ with $q > 2n$. For all $\delta > 0$, there exists $\alpha > 0$ with the following property: If v is subharmonic and $v_1, \ldots, v_q$ are δ-subharmonic in $D(2)$ with Riesz measures (charges) ν, and $\nu_1, \ldots, \nu_q$ satisfying:

$$\begin{aligned}
\left(\nu + \sum_{j=1}^{q} \nu_j\right)(\overline{D(1)}) &\leq M \\
\left|\max_{j \in I} v_j - v\right| &\leq \alpha \quad \text{for all } I \text{ s.t. } |I| = n+1, \\
\text{and} \quad \sum_{j=1}^{q} \nu_j^{-}(\overline{D(1)}) &\leq \alpha, \quad \text{where here } \nu_j = \nu_j^{+} - \nu_j^{-},
\end{aligned}$$

then the signed measure $\kappa = \sum_{j=1}^{q} \nu_j - (q-2n)\nu$ satisfies $\int \psi \, d\kappa \geq -\delta$ for all continuous functions ψ such that $0 \leq \psi \leq 1$ with support in $D(1)$ and gradient bounded by L.