The Potential Theoretic Method of Eremenko & Sodin Part II (Appendix): Some Additonal Context around the Cherry-Eremenko Work

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Classical Landau Theorem

Theorem (Landau's Theorem (1904))

Let f be analytic on $\mathbf{D}$ omitting 0 and 1. Then, $|f'(0)|$ can be bounded explicitly in terms of $|f(0)|$.
Wan-Tzei Lai's (1979) and Hempel's (1979) formulation :

$$|f'(0)| \leq 2|f(0)| \left(\Big| \log |f(0)| \Big| + \frac{\Gamma(1/4)^4}{4\pi^2} \right)$$

Remark

- *This formulation has the correct asymptotics as $|f(0)| \to 0$ and $\Gamma(1/4)^4/4\pi^2 \approx 4.37$ is sharp.*
- *This formulation is not invariant under the affine involution exchanging 0 and 1.*
  - *Of course one can symmetrize to:*

$$|f'(0)| \leq 2\min\left\{ |f(0)| \left(\Big| \log |f(0)| \Big| + \frac{\Gamma(1/4)^4}{4\pi^2} \right), |f(0)-1| \left(\Big| \log |f(0)-1| \Big| + \frac{\Gamma(1/4)^4}{4\pi^2} \right) \right\}$$

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Spherical Derivative

Definition

If $f = f_1/f_0$ is meromorphic on a domain in $\mathbf{C}$, we define the **spherical derivative** of f, denoted $f^\#$, by

$$f^\#(z) = \frac{|f_0(z)f_1'(z) - f_0'(z)f_1(z)|}{|f_0(z)|^2 + |f_1(z)|^2} = \begin{cases} \dfrac{|f'(z)|}{1+|f(z)|^2} & \text{if } f(z) \neq \infty \\ \left|\left(\dfrac{1}{f}\right)'(z)\right| & \text{if } f(z) = \infty. \end{cases}$$

- $f^\#$ measures infinitesimal length distortion by f when the spherical metric is used on $\mathbf{P}^1$ and the Euclidean metric is used on the domain.
- Bounding $f^\#(0)$ bounds $|f'(0)|$ in terms of $|f(0)|$, so giving a bound on $f^\#(0)$ may be called a "Landau theorem."

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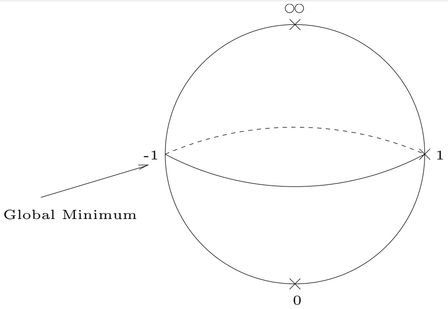
Theorem (Bonk/Cherry (1996))

Let f be analytic on $\mathbf{D}$ omitting 0 and 1. Then,

$$f^{\#}(0) \leq \frac{\Gamma(1/4)^4}{4\pi^2} \quad \text{so} \quad |f'(0)| \leq \frac{\Gamma(1/4)^4}{4\pi^2}(1 + |f(0)|^2).$$

Equality holds if and only if f is the universal covering map such that $f(0) = -1$.



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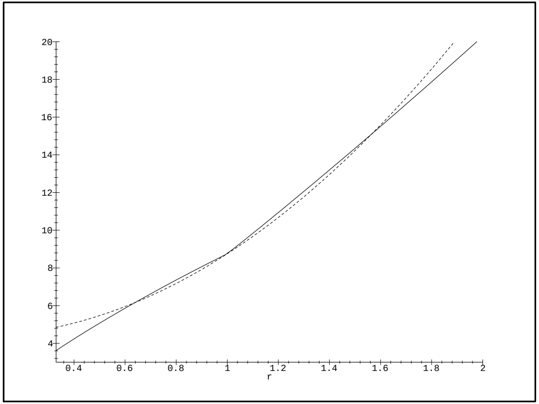

For what domain is the upper bound on $f^{\#}(0)$ the smallest?

Question

Among all domains with exactly four boundary points in the Riemann sphere, is it true that the upper-bound for $f^{\#}(0)$ is smallest for the domain whose boundary points consist of the vertices of a regular tetrahedron inscribed in the Riemann sphere?

Bonk and Cherry were able to use Minda's reflection principle to show that among all domains with exactly three boundary points, the complement of the third roots of unity was the domain with the smallest upper-bound on $f^{\#}(0)$.

Theorem (Solynin 1997)

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What happens as the points coalesce?

Consider $f_r(z) = e^{rz}$ which omits the three points 0, m, and ∞ in $\mathbf{P}^1$, provided $r < \log m$.

- $f_r^\#(0) = \dfrac{r}{2} \to \dfrac{\log m}{2}$.

Let f_1 denote the universal covering map of $\mathbf{C} \setminus \{1, -1\}$ such that $f_1(0) = 0$. Let $f_m(z) = mf_1(z)$, which omits m, $-m$, and ∞ in $\mathbf{P}^1$.

- $f_m^\#(0) = m\dfrac{\Gamma(1/4)^4}{4\pi^2} \approx 4.4m$

Fubini-Study Derivatives and Dufresnoy's Theorem

Definition

For a holomorphic curve $f = (f_0, \ldots, f_n)$ to $\mathbf{P}^n$, the **Fubini-Study derivative** $f^\#$ is defined by

$$(f^\#)^2 = \frac{\sum_{j<k} |f_j f_k' - f_k f_j'|^2}{\left(\sum_j |f_j|^2\right)^2}.$$

Theorem (Dufresnoy (1944))

Let $H_0, \ldots, H_{2n}$ be $2n+1$ hyperplanes in general position in $\mathbf{P}^n$. Then, there exists a constant K depending only on the hyperplanes (and the dimension) such that if

$$f : \mathbf{D} \to \mathbf{P}^n \setminus \{H_0, \ldots, H_{2n}\}, \quad \text{then} \quad f^\#(0) \leq K.$$

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Previous Work

Dufresnoy's proof is based on a normality criterion of Bloch and Cartan, and is hence ineffective. Dufresnoy himself remarked that K depends on the hyperplanes in a completely unknown way.

Goal: Give an explicit estimate for K and investigate its dependence on the hyperplanes.

Previous work in this direction:

P. Hall: An attempt to make pointwise Bloch-Cartan estimates explicit on $\mathbf{P}^2$, but Hall's estimates degenerate where they shouldn't and hence do not give a global estimate.

Cowen: Constructs a negatively curved metric on $\mathbf{P}^n \setminus 2^n + 1$ hyperplanes in general position. The number of hyperplanes was reduced by **Pit-Mann Wong** in his thesis, but not to the optimal $2n + 1$.

Babets: A negative curvature estimate that depends on higher derivatives (or associated curves) of f.

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Notation

- Let $H_j(X_0, \ldots, X_n) = \sum_k a_{jk} X_k$ be linear defining forms for $n+1$ hyperplanes in general position in $\mathbf{P}^n$, normalized so that $||H_j||^2 = \sum_k |a_{jk}|^2 = 1$.
- The general position assumption exactly means the matrix:

$$A = \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & & \vdots \\ a_{n0} & \cdots & a_{nn} \end{pmatrix} \qquad \text{is invertible.}$$

- Hence, AA^* has $n+1$ positive eigenvalues. Denote them by

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Effective Landau Theorem

Let $H_0, \ldots, H_{2n}$ be $2n+1$ hyperplanes in general position in $\mathbf{P}^n$.

- $G = \max_{0 \le i_0 < i_1 < \cdots < i_n \le 2n} \max \left\{ \log \lambda_n, \log \frac{n+1}{\lambda_0} \right\}.$
- $G^{\#} = \min_{0 \le i_0 < i_1 < \cdots < i_n \le 2n} \frac{\lambda_n}{\sqrt{\lambda_0 \lambda_1}}$
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Theorem (Cherry and Eremenko 2006/2010)

If $f : \mathbf{D} \to \mathbf{P}^n$ omits the H_j, then $f^{\#}(0) \le K$, where K can be taken to be

$$K = 12{,}672(2.6 \cdot 10^7 \log B + 10^8)^{6(4 \log B + 20)} \cdot GG^{\#}.$$

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Explicit connection to the geometry of the hyperplanes

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Can one find hyperplane arrangements where G remains bounded as $n \to \infty$?

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A deficiency in this approach is that the estimate does not improve if more than $2n+1$ hyperplanes are deleted.

Schottky type theorem

Theorem (Dufresnoy)

Let

$$\delta_j(z) = \frac{|H_j \circ f(z)|}{||f||},$$

recalling that we have normalized the defining forms of our hyperplanes so that $||H_j|| = 1$. *Then for* $j = 0, \ldots, n$ *and* $|z| < 1$,

$$\log \frac{1}{\delta_j(z)} < \frac{1}{1-|z|}\left[16 \log \frac{1}{\delta_j(0)} + 8K^2\right]$$

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Back to the one-dimensional examples

- $G = \max_{0 \le i_0 < i_1 < \cdots < i_n \le 2n} \max \left\{ \log \lambda_n, \log \frac{n+1}{\lambda_0} \right\}.$
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- $f_r^{\#}(0) = \frac{r}{2} < \frac{\log m}{2}$
- $G^{\#} = 1$
- $G = \log 2 - \log \left(1 - \sqrt{\frac{m^2}{1+m^2}} \right) \approx 2 \log m.$

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- $G = \log 2 - \log \left(1 - \sqrt{\frac{m^2}{1+m^2}} \right) \approx 2 \log m.$

Back to the one-dimensional examples

- $G = \max_{0 \le i_0 < i_1 < \cdots < i_n \le 2n} \max \left\{ \log \lambda_n, \log \frac{n+1}{\lambda_0} \right\}.$
- $G^{\#} = \min_{0 \le i_0 < i_1 < \cdots < i_n \le 2n} \frac{\lambda_n}{\sqrt{\lambda_0 \lambda_1}}$
- $K = 12{,}672 (2.6 \cdot 10^7 \log B + 10^8)^{6(4 \log B + 20)} \cdot G\, G^{\#}.$

Let f_1 denote the universal covering map of $\mathbf{C} \setminus \{1, -1\}$ such that $f_1(0) = 0$. Let $f_m(z) = m f_1(z)$, which omits m, $-m$, and ∞ in $\mathbf{P}^1$.

- $f_m^{\#}(0) = m \frac{\Gamma(1/4)^4}{4\pi^2} \approx 4.4m$
- $G \approx 2 \log m$
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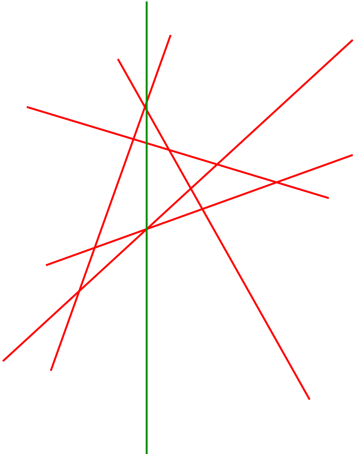
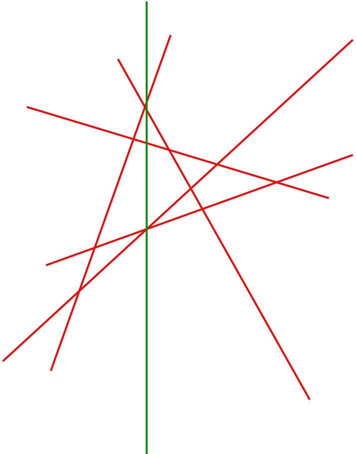
Five lines in $\mathbf{P}^2$



Five lines in $\mathbf{P}^2$

Question

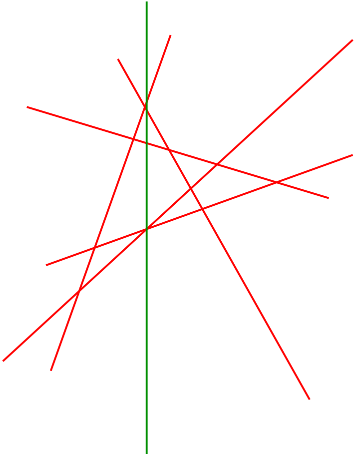
Is there a constant D such that $f^{\#}(0) > D$ implies that the image of f is contained in a diagonal line?



Five lines in $\mathbf{P}^2$

Question

Can we reinterpret the constants G and $G^{\#}$ in terms of the distances between the three points where the diagonal lines intersect the omitted hyperplanes?

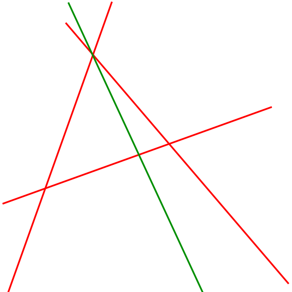


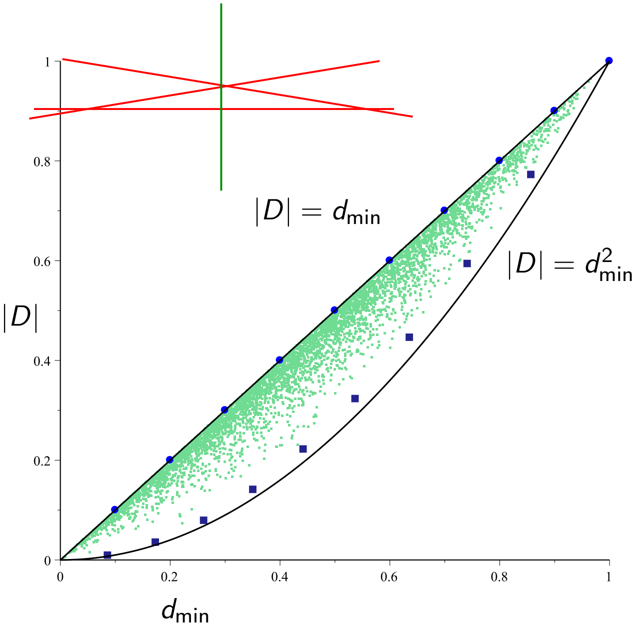
The Linear Algebra of Projective Simplices

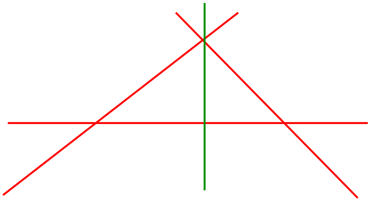
Theorem (Fincher/Olney/Cherry – *Involve* 8 (2015), 707–719)

Let $\mathbf{u}_0, \ldots, \mathbf{u}_n$ be $n+1$ linearly independent unit vectors in $\mathbf{C}^{n+1}$ representing $n+1$ linear forms defining $n+1$ hyperplanes in general position in $\mathbf{P}^n$, which we think of as the faces of a projective simplex. For each j from 0 to n, let d_j denote the Fubini-Study distance from the hyperplane represented by $\mathbf{u}_j$ to the the opposite vertex of the simplex. Let $d_{\min}$ denote the minimum of the d_j. Then,

$$d_{\min}^n \le D = |\det(\mathbf{u}_0, \ldots, \mathbf{u}_n)| \le d_{\min}.$$




$|D| = d_{\min}$
$|D| = d_{\min}^2$
$|D|$
$d_{\min}$
0
0.2
0.4
0.6
0.8
1



A Conjecture

Conjecture

Fix $0 < D \leq 1$ *and consider all configurations of* $\mathbf{u}_0, \ldots, \mathbf{u}_n$ *such that* $D = |\det(\mathbf{u}_0, \ldots, \mathbf{u}_n)|$. *Among all such configurations, the configuration with the largest* d_{min} *will be a regular simplex.*

Remark

When $D < 1$, *we see* $d_{\text{min}}^n < D$ *for the regular simplex with determinant* D.

Proof of the linear algebra result

Proposition

Let $\mathbf{a}, \mathbf{b}_1, \ldots, \mathbf{b}_n$ *be* $n+1$ *linearly independent unit vectors in* $\mathbf{C}^{n+1}$ *representing* $n+1$ *points in general position in* $\mathbf{CP}^n$*. Then, the Fubini-Study distance* d *from the point* $\mathbf{a}$ *to the hyperplane in* $\mathbf{CP}^n$ *spanned by* $\mathbf{b}_1, \ldots, \mathbf{b}_n$ *is given by*

$$d = \frac{|\det(\mathbf{a}, \mathbf{b}_1, \ldots, \mathbf{b}_n)|}{|\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_n|}.$$

This gives the straightforward estimate $|D| \leq d_{\min}$. The more interesting $d_{\min}^n \leq |D|$ follows from a higher dimensional generalization of Lagrange's formula that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

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Geometric Interpretation

- Our result is a global lower bound on the ratio of the infitesimal Kobayashi metric (*i.e.*, the Royden function) on the hyperplane complement divided by the Fubini-Study metric.

Goals for Future Research

- Rather than expressing the geometric constants G and $G^{\#}$ in terms of the eigenvalues of matrices formed by the coefficients of the linear forms defining the hyperplanes, express them in a more geometric way, for example in terms of the distance between the common intersection points of the hyperplanes.
- Can we get a pointwise estimate that tells us how the Kobayashi metric changes as we move around the hyperplane complement?
- Can the estimate be extended to hypersurface complements, rather than only hyperplane complements?

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