The Potential Theoretic Method of Eremenko & Sodin Part II: An Effective Landau theorem for Holomorphic Curves in Projective Space

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- W. Cherry and A. Eremenko, Landau's Theorem for Holomorphic Curves in Projective Space and The Kobayashi Metric on Hyperplane Complements, *Pure and Applied Mathematics Quarterly* **7** (2010), 199–221.

Bounding the amount of area covered

Theorem (Cherry & Eremenko)

Let f be a holomorphic map from the unit disc $|z| < 1$ to $\mathbf{P}^n$ omitting $2n+1$ hyperplanes $H_0, \ldots, H_{2n}$ in general position. Then,

$$\int_{|z|<\frac{1}{32}} f^*\omega_{FS} \leq 36 \left(2.6 \cdot 10^7 \log B + 10^8\right)^{6(4 \log B + 20)} G,$$

where B is the binomial coefficient

$$B = \binom{2n+1}{n+1}$$

and G is an explicit constant that can be determined from the coefficients of the defining forms of the hyperplanes.

The steup

- As in the first lecture, let $(f_0 : \cdots : f_n)$ be projective coordinate functions for f, let

$$u = \log(|f_0|^2 + \cdots + |f_n|^2), \text{ and let } u_j = \log |H_j \circ f|^2.$$

  This time we also must normalize the linear forms H_j so that the maximum of the absolute value of the coefficients is one.
- As before, u is subharmonic, but not harmonic, and the u_j are harmonic.
- The general position hypothesis again implies that

$$\max_{j \in I} u_j = u + O(1) = \max_{0 \le j \le 2n} u_j + O(1),$$

  where the $O(1)$ terms can be explicitly determined and result in the consant G in the theorem.

By the Rickman Covering Lemma, for $c > 1$, there exists a and r such that

$$\mu(D(a, 16r)) \leq 16^4 c\mu(D(a, r))$$

and such that

$$\sigma_{a,r} \stackrel{\text{def}}{=} \mu(D(a, r)) \geq \frac{1}{c}\mu(D(0, 1/32)),$$

where as before, μ is the Riesz measure of u.

Let $\tilde{u}$ be the least harmonic majorant of $\max\limits_{0 \leq j \leq 2n} u_j$ and re-scale only once this time by setting

$$v(z) = \frac{1}{\sigma_{a,r}}[u(a + rz) - \tilde{u}(a + rz)] \qquad \text{and} \qquad v_j(z) = \frac{1}{\sigma_{a,r}}[u_j(a + rz) - \tilde{u}(a + rz)].$$

Then v_j are harmonic and negative on $|z| \leq 16$ with $\max v_j(z) = 0$ when $|z| = 16$. The function v is subharmonic on $|z| \leq 16$ with

$$\int\limits_{|z| \leq 1} dd^c v = 1 \qquad \text{and} \qquad \int\limits_{|z| \leq 16} dd^c v \leq 16^4 c.$$

Now, for $|z| \leq 16$ and any index set I of cardinality $n+1$,

$$\left| v - \max_{j \in I} v_j \right| \leq \varepsilon, \qquad \text{where } \varepsilon \stackrel{\text{def}}{=} \frac{G}{\sigma_{a,r}}.$$

Since the v_j are negative, this also tells me that $v(z) \leq \varepsilon$ for $|z| \leq 16$ and $v(z) \geq -\varepsilon$ when $|z| = 16$.

Let $U_I = \{z : |z| \leq 2 \text{ and } |v(z) - v_j(z)| < \varepsilon \text{ for all } j \in I\}$.

As before $\bigcup_I U_I$ is the whole disc, and so ther eis some index set I_0 such that the area of U_{I_0} is at least $\frac{4\pi}{B}$. Without loss of generality, we may assume $I_0 = \{0, \ldots, n\}$ and we may shrink U so that the area U is exactly $\frac{4\pi}{B}$.

Using Jensen to Move to a smaller disk

Recall $v(z) \geq -\varepsilon$ when $|z| = 16$. By the Jensen Formula

$$\frac{1}{2}\left[\int_0^{2\pi} v(16e^{i\theta})\frac{d\theta}{2\pi} - \int_0^{2\pi} v(2e^{i\theta})\frac{d\theta}{2\pi}\right] = \int_2^{16} \frac{dt}{t} \int\limits_{|z|\leq t} dd^c v \leq 16^4 c \log 8,$$

and we conclude that there is a point z' with $|z'| = 2$ such that

$$v(z') \geq -2 \cdot 16^4 c \log 8 - \varepsilon.$$

Cartan's Lemma

Theorem (Cartan)

Let μ be a finite Borel measure on $|z| < 1$, and let Φ be the Blaschke potential

$$\Phi(z) = \int_{|\zeta|<1} \log \left| \frac{z-\zeta}{1-\bar{\zeta}z} \right| d\mu(\zeta).$$

Let $0 < r < 1$, let $\eta < 1/4e$, and let $|z_0| \leq r$. Then, there is a countable family of exceptional discs D_j, the sum of whose radii does not exceed $4e\eta$, such that for those z with $|z| \leq r$ and not in any of the D_j, we have

$$\Phi(z) > C(r,\eta)\Phi(z_0),$$

where

$$C(r,\eta) = \frac{4}{(1-r)^2} \log \frac{1}{\eta}.$$

We can use the Cartan Lemma to ensure that we find a point z_0 in our set U so that $v(z_0)$ is not too small, namely,

$$v(z_0) \geq C(1/8, \eta)[-2 \cdot 16^4 c \log 8 - 2\varepsilon].$$

Since v_j are ϵ close to v for $j = 0, \ldots, n$, we also get a lower bound on $v_j(z_0)$.

Tying things together

Since the v_j are negative harmonic functions, we can then use the Harnack inequality (twice) to give a lower bound on $v_j(z)$ *for all* $|z| \leq 12$.

Now suppose there were some δ so that $|v_j(z) - v_k(z)| \leq \delta$ for $|z| \leq 2$ and $0 \leq j, k \leq n$. Because at each point, at least one of the v_j come within ε of v this would tell us that $|v - v_j| \leq \delta + \varepsilon$ for all $|z| \leq 2$. Since we know the Riesz mass of v is not too small, the Jensen formula tells us that the harmonic v_j cannot be too close to the non-harmonic v, and so $\varepsilon + \delta \geq \log 2$. Thus, if ε is very small, then δ is not.

Finally, since v_j and v_k are ε close on the positive measure set U, one can use what's known as **propogation of smallness** to conclude they cannot be very far apart anywhere. Thus, if ε were very small, δ would have to be small as well, which we saw cannot happen. This results in a lower bound on ε, which since $\varepsilon \stackrel{\text{def}}{=} \dfrac{G}{\sigma_{a,r}}$, gives an upper bound $\sigma_{a,r}$ and hence also $\mu(D(0, 32)$ as desired.

Propogation of Smallness

Theorem (Three Circles Theorem)

Let u be a harmonic function in the unit disc with $|u| < 1$. Let $0 < r < R < 1/2$ and assume $|u(z)| < r^\tau$ for $|z| < r$ and some $\tau > 0$. Then, $|u(z)| < 2(2R)^\tau$ for $|z| < R$.

Proof.

The classical Hadamard three circles theorem implies, $\int_0^{2\pi} [u(\rho e^{i\theta})]^2 \frac{d\theta}{2\pi} \leq \rho^{2\tau}$ $r < \rho < 1$.

Thus, $\int_r^{2R} 2\rho \int_0^{2\pi} [u(\rho e^{i\theta})]^2 \frac{d\theta}{2\pi} d\rho \leq \frac{(2R)^{2\tau+2} - r^{2\tau+2}}{\tau+1} \leq (2R)^{2\tau+2} - r^{2\tau+2}$.

Trivially, $\int_0^r 2\rho \int_0^{2\pi} [u(\rho e^{i\theta})]^2 \frac{d\theta}{2\pi} d\rho \leq r^{2\tau+2}$.

Proof of three circles continued

Continuation of the proof.

Let z_0 be a point with $|z_0| = R$ such that $|u(z)| \leq |u(z_0)|$ for $|z| \leq R$. Then, by Jensen's Theorem,

$$\begin{aligned}(2R)^{2\tau+2} &\geq \int_0^{2R} 2\rho \int_0^{2\pi} [u(\rho e^{i\theta})]^2 \frac{d\theta}{2\pi} d\rho \\ &\geq \int_0^{R} 2t \int_0^{2\pi} [u(z_0 + te^{i\theta})]^2 \frac{d\theta}{2\pi} dt \\ &\geq \int_0^{R} 2t \left[\int_0^{2\pi} u(z_0 + te^{i\theta}) \frac{d\theta}{2\pi}\right]^2 dt = [u(z_0)]^2 R^2. \qquad \square\end{aligned}$$

Nadirashvili

Theorem (Nadirashvili)

Let u be a harmonic function in the unit disc such that $|u| < 1$. Let $0 < \rho < 1/5$. Let U be a subset of $D(0, \rho)$ with area $\alpha\pi\rho^2 > 0$. Let

$$r = \frac{1}{2}\left(\frac{\alpha}{36e}\right)^2,$$

and assume $r < \rho/2$. If $|u(z)| < \varepsilon < 1/9$ for all z in U, then there is a point z_0 in $D(0, \rho)$ such that

$$|u(z)| < 3\sqrt{\varepsilon}, \qquad \text{for all } z \in D(z_0, r). \tag{1}$$

Moreover,

$$\log|u(z)| < \frac{\log(9\varepsilon)\log(1/(5\rho))}{2\log\left[\frac{8}{5}\left(\frac{36e}{\alpha}\right)^2\right]} + \log 2 \qquad \text{for all } |z| < \rho. \tag{2}$$

Fubini-Study Derivatives

Definition

For a holomorphic curve $f = (f_0, \ldots, f_n)$ to $\mathbf{P}^n$, the **Fubini-Study derivative** $f^\#$ is defined by

$$(f^\#)^2 = \frac{\sum_{j<k} |f_j f_k' - f_k f_j'|^2}{\left(\sum_j |f_j|^2\right)^2}.$$

From an area bound to a derivative bound

Proposition

Let f be a holomorphic function without zeros on $|z| < 1$ such that $|f(z)| < 1$. Then,

$$\left|\frac{f'(0)}{f(0)}\right| \leq -\log|f(0)|^2.$$

Proof.

Let $r < 1$. Since $\dfrac{f'(z)}{f(z)} = \dfrac{\partial}{\partial z}\log|f(z)|^2$, differentiating the Poisson formula for $\log|f(z)|^2$ gives us

$$\begin{aligned}\left|\frac{f'(0)}{f(0)}\right| &= \frac{1}{r}\left|\int_0^{2\pi} e^{-i\theta}\log|f(re^{i\theta})|^2\frac{d\theta}{2\pi}\right| \\ &\leq \frac{1}{r}\int_0^{2\pi} -\log|f(re^{i\theta})|^2\frac{d\theta}{2\pi} = -\frac{1}{r}\log|f(0)|^2 \quad \text{and let } r \to 1\end{aligned}$$

□

Theorem (Area to Derivative)

Let $f = [f_0 : \ldots : f_n]$ be a holomorphic map from the unit disc $|z| < 1$ to $\mathbf{P}^n$ which omits the $n+1$ coordinate hyperplanes. If

$$\int_{|z|<1} (f^{\#})^2 \frac{dA}{\pi} \leq \sigma,$$

then

$$f^{\#}(0) \leq 3\sqrt{2}[(2\log 2)\sigma + \log(n+1)] < 5.89\sigma + 4.25\log(n+1).$$

The proof is essentially the proposition and the Harnack inequality.

Remark

When $n = 1$, the example $f(z) = az$ shows the hypothesis about missing the coordinate hyperplanes cannot be omitted.

Effective Landau Theorem

Let $H_0, \ldots, H_{2n}$ be $2n+1$ hyperplanes in general position in $\mathbf{P}^n$.

Theorem (Cherry and Eremenko)

If $f : \mathbf{D} \to \mathbf{P}^n$ omits the H_j, then $f^{\#}(0) \leq K$, where K can be taken to be

$$K = 14{,}000(2.6 \cdot 10^7 \log B + 10^8)^{6(4 \log B + 20)} \cdot GG^{\#}.$$

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$$K = \underbrace{14{,}000(2.6 \cdot 10^7 \log B + 10^8)^{6(4 \log B + 20)}}_{\text{a huge constant depending only on dimension}} \cdot GG^{\#}.$$

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A deficiency in this approach is that the estimate does not improve if more than $2n+1$ hyperplanes are deleted.