The Potential Theoretic Method of Eremenko & Sodin Part I: Hyperbolicity of the complement of $2n+1$ hypersurfaces in general position

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- A. Eremenko, A Picard Type Theorem for Holomorphic Curves, *Periodica Mathematica Hungarica* **38** (1999), 39–42.

General Introduction to My Mini-Course

- I will post my lecture slides to `http://wcherry.math.unt.edu/pubs.html`

Broadly speaking, there are three approaches to Value Distribution Theory:

1. The logarithmic derivative method of R. Nevanlinna and H. Cartan.
2. The negative curvature method of F. Nevanlinna, T. Shimizu, and L. Ahlfors.
3. The potential-theoretic method of Eremenko and Sodin.

The first two approaches were developed in the early twentieth century. The third approach was developed around 1990 and is much less well known.

The goal of these three lectures is to introduce the Eremenko & Sodin approach and highlight some of its advantages and disadvantages.

Picard's Theorem

Theorem (Picard's Theorem)

If f is a meromorphic function on $\mathbf{C}$ that omits three values in $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$, then f must be constant. In other words, the complement of three points in $\mathbf{P}^1$ is **hyperbolic.**

Picard's Theorem is a 19^{th}-century theorem. Nevanlinna's theory generalizes the *omitted* values in Picard's Theorem to *deficient* values, *i.e.* values that are not necessarily omitted but which are taken on less than expected. This makes Nevanlinna's theory much more technical than the Picard Theorem.

Eremenko & Sodin introduced their method to handle deficient values, and as such it is rather technical. Eremenko wrote the short *Periodica Math. Hungarica* paper some years later to illustrate the method in the easier context of Picard-type theorems.

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- Three is the smallest integer greater than two.
- The Euler characteristic of $\mathbf{P}^1$ is two.
- One can also understand the two by considering the meromorphic one-form $\frac{dz}{z}$ on $\mathbf{P}^1$. Note that $\frac{dz}{z}$ has two poles, one at $z = 0$ and the other at $z = \infty$. Any meromorphic one-form on $\mathbf{P}^1$ has two poles, counting multiplicity.

Picard Theorem for Hyperplanes in Projective Space

A meromorphic n-form on $\mathbf{P}^n$ has $n+1$ poles.

Theorem (Cartan's Hyperplane Theorem)

A holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n$ which omits $n+2$ hyperplanes in general position must be linearly degenerate.

In fact, with some additional argument, one sees that the dimension of the image can be at most $n/2$ and that omitting additional hyperplanes further lowers the dimension:

Theorem (Dufresnoy)

A holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n$ which omits $n+k$ hyperplanes in general position must be contained in a linear subspace of dimension at most n/k.

Corollary

*A holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n$ which omits $2n+1$ hyperplanes in general position must be contained in a linear subspace of dimension at most $n/(n+1) < 1$, i.e., f must be constant. The complement of $2n+1$ hyperplanes in general position in $\mathbf{P}^n$ is **hyperbolic**.*

What about non-linear hypersurfaces?

Remark

- *Cartan's approach relies on the use of Wronskians, so the linearity of the hyperplanes is essential.*
- *Linearity is also essential in Ahlfors's approach.*
- *These days there are techniques to derive results for non-linear hypersurfaces from the hyperplane theorems à la Corvaja and Zannier and Evertse and Ferretti. But to get hyperbolicity by these techniques, so far, one needs to omit a number of hypersurfaces that is quadratic (i.e. n^2) in n, the dimension of the projective space.*
- *The method of Eremenko & Sodin does not see the linearity so applies to hypersurfaces just as easily as it applies to hyperplanes.*
- *The Eremenko & Sodin method also generalizes to quasiregular mappings in* $\mathbf{R}^n$.

General Position

Let M be a closed subset of $\mathbf{P}^N$. Let n be a positive integer. A collection $H_0, \ldots, H_{2n}$ of $2n+1$ hypersurfaces in $\mathbf{P}^N$ such that for every index set $I \subset \{0, \ldots, 2n\}$ of cardinality $n+1$,

$$M \cap \left(\bigcap_{j \in I} H_j \right) = \emptyset$$

will be called n-**general** with respect to M. Note that this is not standard terminology and not Eremenko's terminology. But, in the case that M is a closed subvariety of pure dimension n, then $H_0, \ldots, H_{2n}$ being n-general with respect to M exactly means that $H_0, \ldots, H_{2n}$ are in general position with respect to M.

Eremenko's Hypersurface Picard Theorem

Theorem (Eremenko)

Let M be a closed subset of $\mathbf{P}^N$ and let $H_0, \ldots, H_{2n}$ be hypersurfaces of $\mathbf{P}^N$ that are n-general with respect to M. Then every holomorphic curve

$$f : \mathbf{C} \to M \setminus \bigcup_{j=0}^{2n} H_j$$

is constant. In other words, $M \setminus \bigcup_{j=0}^{2n} H_j$ is hyperbolic.

Rickman's Covering Lemma

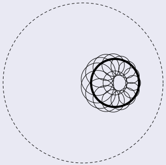
Lemma (**Rickman covering lemma**)

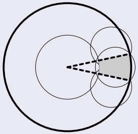
Let μ be a Borel measure on $|z|<1$ such that $|z|<1$ has finite μ-measure. Let $m \geq 1$ be an integer and let $c>1$. Let $\rho_m(b)=\frac{1}{2^{m+1}}(1-|b|)$. Then, there is a complex number a with $|a|<1$ such that

$$\mu(D(a,2^m\rho_m(a))) \leq 16^m c\mu(D(a,\rho_m(a))) \quad \text{and} \quad \mu(D(0,1/2^{m+1})) \leq c\mu(D(a,\rho_m(a))).$$

Proof.

Choose a such that $\mu(D(a,\rho_m(a))) \geq \frac{1}{c}\sup_{|z|<1}\mu(D(z,\rho_m(z)))$.





□

Idea of the Proof

- Assume f is non-constant.
- Choose homogeneous coordinates on $\mathbf{P}^N$ and let $f = (f_0 : \cdots : f_N)$ represent f, where f_j are entire functions without common zeros. Let

$$u = \log ||f||^2 = \log(|f_0|^2 + \cdots + |f_N|^2).$$

  Note that u is subharmonic, and not harmonic since f is non-constant.
- By abuse of notation, let H_j also denote a homogeneous polynomials in $N+1$ variables defining the hypersurfaces H_j. Let d_j be the degree of H_j. Let $u_j = d_j^{-1} \log |H_j \circ f|^2$, which are harmonic since f omits the H_j.
- The fact that M is compact and the n-general assumption tells us that for any $I \subset \{0, \ldots, 2n\}$ of cardinality $n+1$,

$$\max_{j \in I} u_j = u + O(1).$$

- Now, since we have $2n+1$ harmonic functions u_j, this tells us that at any point of $\mathbf{C}$, $n+1$ of the u_j are close to u and hence also close to each other.
- This implies the non-harmonic u is close to a harmonic u_j on a large portion of the complex plane, and this will be a contradiction.

Proof.

- Let μ be the Riesz measure of u, *i.e.* $\mu(E) = \int_E \Delta u \frac{dA}{2\pi}$.
- Note that since $T_f(r) = \int_0^r \frac{dt}{t}\mu(D(0,t))$, $\mu(\mathbf{C}) = \infty$ since f is non-constant.
- Let $R_k \to \infty$. By the Rickman Lemma (with $m = 1$ and $c = 2$), we can find a_k and r_k such that

$$M_k = \mu(D(a_k, r_k)) \geq \frac{1}{2}\mu(D(0, R_k/4)) \to \infty \text{ and } \mu(D(a_k, 2r_k)) \leq 32\mu(D(a_k, r_k)).$$

- Let $\tilde{u}$ denote the smallest harmonic majorant of u on $D(a_k, 2r_k)$. Define

$$u_k(z) = \frac{1}{M_k}[u(a_k + r_k z) - \tilde{u}(a_k + r_k z)] \qquad \text{and}$$

$$u_{j,k}(z) = \frac{1}{M_k}[u_j(a_k + r_k z) - \tilde{u}(a_k + r_k z)].$$

The u_k are subharmonic on $D(0,2)$ and the $u_{k,j}$ are harmonic on $D(0,2)$.

Proof.

- After taking a subsequence, we can assume $u_k \to v$, a sub-harmonic function and $u_{j,k} \to v_j$, harmonic functions (or possibly identically $-\infty$) such that

$$\max_{j \in I} v_j = v$$

  for any $I \subset \{0, \ldots, 2n\}$ of cardinality $n+1$.
- Note that v is not harmonic since by construction ν, the Riesz measure of v is such that $\nu(D(0,1)) \geq 1$.
- For such an I, let $E_I = \{z \in D(0,2) : v(z) = v_j(z) \text{ for all } j \in I\}$.
- Then, $\bigcup_I E_I = D(0,2)$, and so there exists I_0 such that E_{I_0} has positive area.
- For $i, j \in I_0$, the functions v_i and v_j are harmonic functions agreeing on a set of positive area, and hence $v_i = v_j$ everywhere by uniqueness.
- Since $v = \max_{j \in I_0} v_j = v_j$, we see v is also harmonic, a contradiction. □

Summary

- If f is non-constant, we have a non-harmonic subharmonic function u and $2n+1$ harmonic functions u_j on $\mathbf{C}$. The n-general intersection hypothesis lets us conclude that for each index set of cardinality $n+1$,

$$\max_{j \in I} u_j = u + O(1) = \max_{0 \le j \le 2n} u_j + O(1).$$

- The Rickman covering lemma let's us re-scale in such a way to kill the $O(1)$ terms and get harmonic v_j and non-harmonic subharmonic v on $D(0,2)$ such that

$$\max_{j \in I} v_j = v = \max_{0 \le j \le 2n} v_j.$$

- Then, v agrees with $n+1$ of the v_j on a set of positive area, and hence $n+1$ of the v_j are identical and must also equal v everywhere, a contradiction since v is not harmonic.
- In my next lecture, I want to start with functions on a disc instead of $\mathbf{C}$, and so we cannot re-scale to kill the $O(1)$ terms.

Some disadvantages of this approach

- The method does not "see" the degree of the hypersurfaces H_j. As the degree of the H_j increases, one should need fewer of them to get hyperbolicity of the complement.
- Recall Dufresnoy's Theorem that a holomorphic curve in the complement of $n+k$ hyperplanes in general position in $\mathbf{P}^n$ lies in a linear subspace of dimension at most n/k. This is a "degeneracy" theorem rather than a hyperbolicity theorem. This method does not seem to be able to prove degeneracy.