Lectures on Non-Archimedean Function Theory
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Lecture 1: Analogs of Basic Complex Function Theory

Lecture 2: Valuation Polygons and a Poisson-Jensen Formula

Lecture 3: Non-Archimedean Value Distribution Theory

Lecture 4: Benedetto's Non-Archimedean Island Theorems

This lecture series is an introduction to non-Archimedean function theory. The audience is assumed to be familiar with non-Archimedean fields and non-Archimedean absolute values, as well as to have had a standard introductory course in complex function theory. A standard reference for the later could be, for example, [Ah 2]. No prior exposure to non-Archimedean function theory is supposed. Full details on the basics of non-Archimedean absolute values and the construction of p-adic number fields, the most important of the non-Archimedean fields, can be found in [Rob].

1 Analogs of Basic Complex Function Theory

1.1 Non-Archimedean Fields

Let A be a commutative ring. A **non-Archimedean absolute value** $|\ |$ on A is a function from A to the non-negative real numbers $\mathbf{R}_{\geq 0}$ satisfying the following three properties:

AV 1. $|a| = 0$ if and only if $a = 0$;

AV 2. $|ab| = |a| \cdot |b|$ for all $a, b \in A$; and

AV 3. $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in A$.

Exercise 1.1.1. *Prove that* **AV 3** *implies that if* $|a| \neq |b|$, *then* $|a + b| = \max\{|a|, |b|\}$.

Remark. There are two geometric interpretations of Exercise 1.1.1. The first is that every triangle in a non-Archimedean world is isosceles. The second is that every point inside a circle may serve as a center of the circle. This also means that either two discs are disjoint or one is contained inside the other.

Exercise 1.1.2. *If* $|\ |$ *is a non-Archimedean absolute value on an integral domain* A, *prove that* $|\ |$ *extends uniquely to the fraction field of* A.

A pair $(\mathbf{F}, |\ |)$ consisting of a field $\mathbf{F}$ together with a non-Archimedean absolute value $|\ |$ on $\mathbf{F}$ will be referred to as a **non-Archimedean field** and denoted simply by $\mathbf{F}$ for brevity. A sequence a_n in a non-Archimedean field $\mathbf{F}$ is said to **converge** to an element a in $\mathbf{F}$, if for every $\varepsilon > 0$, there exists a natural number N such that for all natural numbers $n \geq N$, we have $|a_n - a| < \varepsilon$. A sequence a_n in $\mathbf{F}$ is called a **Cauchy sequence** if for every $\varepsilon > 0$, there exists a natural number N such that if m and n are both natural numbers $\geq N$, then $|a_n - a_m| < \varepsilon$. As in elementary analysis, it is easy to see that every convergent sequence is Cauchy. In general, not every Cauchy sequence must converge. However, if the non-Archimedean field $\mathbf{F}$ is such that every Cauchy sequence of elements in $\mathbf{F}$ converges, then $\mathbf{F}$ is called **complete**.

Exercise 1.1.3. *Let* $\mathbf{F}$ *be a non-Archimedean field. Let* $\overline{\mathbf{F}}$ *be the set of Cauchy sequences in* $\mathbf{F}$ *modulo the sequences which converge to* 0. *In other words, define an equivalence relation on the set of Cauchy sequences in* $\mathbf{F}$ *by defining two Cauchy sequences to be equivalent if their difference is a sequence which converges to* 0, *and let* $\overline{\mathbf{F}}$ *be the set of equivalence classes under this equivalence relation. Then, show* $\overline{\mathbf{F}}$ *is field, that* $|\ |$ *naturally extends to* $\overline{\mathbf{F}}$, *and that* $\overline{\mathbf{F}}$ *is a complete non-Archimedean field, which we call the* ***completion*** *of* $\mathbf{F}$.

Given a field $\mathbf{F}$, we use $\mathbf{F}^\times$ to denote $\mathbf{F} \setminus \{0\}$. Given a non-Archimedean field $(\mathbf{F}, |\ |)$, the set $|\mathbf{F}^\times| = \{|x| : x \in \mathbf{F}^\times\} \subset \mathbf{R}_{>0}$ is a subgroup under multiplication of $\mathbf{R}_{>0}$ and is called the **value group** of $\mathbf{F}$. If $|\mathbf{F}^\times|$ is discrete in $\mathbf{R}_{>0}$, then $\mathbf{F}$ is called a **discretely valued** non-Archimedean field.

We now present some fundamental examples of non-Archimedean fields.

The Trivial Absolute Value

Let $\mathbf{F}$ be any field. Define an absolute value $|\ |$, called the **trivial absolute value**, on $\mathbf{F}$ by declaring that $|0| = 0$ and $|x| = 1$ for all x in $\mathbf{F}^\times$. Clearly a sequence is Cauchy with respect to the trivial absolute value if and only if it is eventually constant, and hence convergent. Thus any field can be made into a complete non-Archimedean field by endowing it with the trivial absolute value.

p-Adic Number Fields

Consider the rational numbers $\mathbf{Q}$, and let p be a prime number. Then, any non-zero x in $\mathbf{Q}$ can be written as

$$x = p^n \frac{a}{b},$$

where p does not divide a or b. If we define $|x|_p = p^{-n}$ and $|0|_p = 0$, then we easily see that $|\ |_p$ is a non-Archimedean absolute value on $\mathbf{Q}$.

Exercise 1.1.4. *Let p be a prime number, let n_0 be an integer, and for each integer $n \geq n_0$, let a_n be an integer between zero and $p-1$, inclusive. Show that sequence of partial sums*

$$S_k = \sum_{n=n_0}^{k} a_n p^n$$

is a Cauchy sequence in $(\mathbf{Q}, |\ |_p)$. Moreover, show that S_k converges in $\mathbf{Q}$ if and only if the a_n are eventually periodic, or in other words there exists integers n_1 and $t \geq 1$ such that $a_{n+t} = a_n$ for all $n \geq n_1$. Hint: A solution can be found in [Rob, §I.5.3].

We conclude from Exercise 1.1.4 that $\mathbf{Q}$ is not complete with respect to $|\ |_p$, because, for example, the sequence of partial sums

$$S_k = \sum_{n=0}^{k} p^{n^2}$$

is Cauchy, but not convergent. We denote by $\mathbf{Q}_p$ the completion of $\mathbf{Q}$ with respect to $|\ |_p$ and call this field the field of p**-adic numbers**. The closure of the integers $\mathbf{Z}$ in $\mathbf{Q}_p$ is denoted by $\mathbf{Z}_p$, and elements of $\mathbf{Z}_p$ are called p**-adic integers**.

Exercise 1.1.5. *Fix a prime number p. Every non-zero element x in $\mathbf{Q}_p$ has a unique p-adic expansion of the form*

$$x = \sum_{n=n_0}^{\infty} a_n p^n,$$

where the a_n are integers between 0 and $p-1$, $a_{n_0} \neq 0$, and $p^{-n_0} = |x|_p$.

Exercise 1.1.6. *Finite algebraic extensions of complete non-Archimedean fields are again complete non-Archimedean fields. Hint: See* [Lang, Ch. XII].

Finite algebraic extensions of $\mathbf{Q}_p$ are called p**-adic number fields**.

Exercise 1.1.7. *No p-adic number field is algebraically closed. Hint: Show that the value group of any finite extension of $\mathbf{Q}_p$ must be discrete and hence cannot contain all the n-th roots of p for all n.*

Theorem 1.1.8. *The absolute value $|\ |_p$ extends uniquely to the algebraic closure $\mathbf{Q}_p^a$ of $\mathbf{Q}_p$, which is not complete, but its completion $\mathbf{C}_p$ remains algebraically closed.*

The field $\mathbf{C}_p$ is called the p**-adic complex numbers**. I will not discuss the proof of Theorem 1.1.8 here. See [Rob, Ch. III] for a proof.

Positive Characteristic

The following fields are important positive characteristic analogs of the p-adic number fields. Let $\mathbf{F}_q$ denote the finite field of q elements, where q is a power of a prime. Let $\mathbf{F}_q(T)$ denote the field of rational functions over $\mathbf{F}_q$. In positive characteristic number theory, the field $\mathbf{F}_q(T)$ plays the role of the rational numbers and the polynomial ring $\mathbf{F}_q[T]$ plays the role of the integers. The notation $|\ |_\infty$ is often used to denote the unique non-Archimedean absolute value on $\mathbf{F}_q(T)$ such that $|T|_\infty = q$. The completion of $\mathbf{F}_q(T)$ with respect to $|\ |_\infty$ is isomorphic to $\mathbf{F}_q((1/T))$, the formal Laurent series ring in $1/T$ with coefficients in $\mathbf{F}_q$. The complete non-Archimedean field $(\mathbf{F}_q((1/T)), |\ |_\infty)$ is a positive characteristic analog of the p-adic number fields, namely the finite extensions of $\mathbf{Q}_p$. As with the p-adic number fields, the absolute value $|\ |_\infty$ extends uniquely to the algebraic closure of $\mathbf{F}_q((1/T))$, and the completion of $\mathbf{F}_q((1/T))^a$ remains algebraically closed, and is denoted by $\mathbf{C}_\infty$, or possibly $\mathbf{C}_{p,\infty}$ if one wants to also emphasize the characteristic. Hence $\mathbf{C}_\infty$ is a positive characteristic and non-Archimedean analog of the complex numbers.

These notes will discuss analysis over complete algebraically closed non-Archimedean fields. The most important examples of such fields are the fields of p-adic complex numbers $\mathbf{C}_p$ and the fields $\mathbf{C}_\infty$ introduced above. However, rarely is the precise form of the field important, and henceforth $\mathbf{F}$ will denote simply a complete non-Archimedean field. Sometimes we may need to assume that $\mathbf{F}$ has characteristic zero.

1.2 Analytic and Meromorphic Functions

Let $(\mathbf{F}, |\ |)$ be a complete, algebraically closed, non-Archimedean field.

Exercise 1.2.1. *A series $\sum a_n$ of elements of $\mathbf{F}$ converges if and only if $\lim_{n\to\infty} |a_n| = 0$.*

Because of Exercise 1.2.1, there is no need in non-Archimedean analysis for any of the various convergence tests one learns in freshmen calculus.

The formal power series ring $\mathbf{F}[[z]]$ in the variable z with coefficients in $\mathbf{F}$ forms an integral domain with addition and multiplication defined in the natural way. Because of Exercise 1.2.1, an element

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbf{F}[[z]]$$

is seen to converge at the point z in $\mathbf{F}$ if

$$\lim_{n\to\infty} |a_n||z|^n = 0.$$

If a formal power series f converges at z, then clearly f converges at each w with $|w| < |z|$. Similarly, if f diverges at z, then f diverges at each w with $|w| > |z|$. We therefore define the **radius of convergence** r_f of a formal power series f by

$$r_f = \sup\{|z| : f \text{ converges at } z\}.$$

One then has the usual Hadamard formula for the radius of convergence of a formal power series we are familiar with from real or complex analysis.

Exercise 1.2.2 (Hadamard Formula). $r_f = \dfrac{1}{\limsup\limits_{n\to\infty} |a_n|^{1/n}}$.

It is also easy to see that radius of convergence behaves well under addition and multiplication:

Exercise 1.2.3. $r_{f+g} \geq \min\{r_f, r_g\}$ *and* $r_{fg} \geq \min\{r_f, r_g\}$.

Define the **open** or **unbordered ball** of radius R by

$$\mathbf{B}_{<R} = \{z \in \mathbf{F} : |z| < R\}.$$

We also use the notation $\mathbf{B}_{<\infty} = \mathbf{F}$ to include the case of all of $\mathbf{F}$. The **closed** or **bordered ball** of radius $R < \infty$ is defined by

$$\mathbf{B}_{\leq R} = \{z \in \mathbf{F} : |z| \leq R\}.$$

If $R > 0$, then both $\mathbf{B}_{<R}$ and $\mathbf{B}_{\leq R}$ are both open and closed in the topology on $\mathbf{F}$. Because of this some people prefer the somewhat more cumbersome "unbordered" and "bordered" terminology. The ring of **analytic functions** on $\mathbf{B}_{\leq R}$, denoted $\mathcal{A}[R]$, is defined by

$$\mathcal{A}[R] = \left\{ \sum_{n=0}^{\infty} a_n z^n \in \mathbf{F}[[z]] : \lim_{n\to\infty} |a_n| R^n = 0 \right\}.$$

Similarly, the ring of analytic functions on $\mathbf{B}_{<R}$, denoted $\mathcal{A}(R)$, is defined by

$$\mathcal{A}(R) = \left\{ \sum_{n=0}^{\infty} a_n z^n \in \mathbf{F}[[z]] : \lim_{n\to\infty} |a_n| r^n = 0 \text{ for all } r < R \right\}.$$

Elements of $\mathcal{A}(\infty)$, *i.e.* power series with infinite radius of convergence, are called **entire** functions.

All this extends easily to convergent Laurent series. Namely, we can consider various types of bordered, unbordered, or semi-bordered annuli:

$$A[r_1, r_2] = \{z \in \mathbf{F} : r_1 \leq |z| \leq r_2\} \qquad A(r_1, r_2] = \{z \in \mathbf{F} : r_1 < |z| \leq r_2\}$$
$$A[r_1, r_2) = \{z \in \mathbf{F} : r_1 \leq |z| < r_2\} \qquad A(r_1, r_2) = \{z \in \mathbf{F} : r_1 < |z| < r_2\},$$

and the various rings of analytic functions on those spaces

$$\mathcal{A}[r_1, r_2] = \left\{ \sum_{n=-\infty}^{\infty} a_n z^n : \lim_{|n|\to\infty} |a_n| r^n = 0 \text{ for all } r_1 \leq r \leq r_2 \right\}$$
$$\mathcal{A}(r_1, r_2] = \left\{ \sum_{n=-\infty}^{\infty} a_n z^n : \lim_{|n|\to\infty} |a_n| r^n = 0 \text{ for all } r_1 < r \leq r_2 \right\}$$
$$\mathcal{A}[r_1, r_2) = \left\{ \sum_{n=-\infty}^{\infty} a_n z^n : \lim_{|n|\to\infty} |a_n| r^n = 0 \text{ for all } r_1 \leq r < r_2 \right\}$$
$$\mathcal{A}(r_1, r_2) = \left\{ \sum_{n=-\infty}^{\infty} a_n z^n : \lim_{|n|\to\infty} |a_n| r^n = 0 \text{ for all } r_1 < r < r_2 \right\}.$$

Notice that all the above rings of analytic functions are integral domains. Elements of their fraction fields are called **meromorphic functions**. Thus, I will use, for instance $\mathcal{M}(r_1, r_2]$ to denote the fraction field of $\mathcal{A}(r_1, r_2]$, which is the field of meromorphic functions on $A(r_1, r_2]$.

For the most part, I will leave a discussion of analytic and meromorphic functions on subsets of $\mathbf{F}$ more complicated than annuli to the other lecturers in this school, and in particular I refer the reader to Berkovich's lectures.

Remark 1.2.4. If the absolute value $|\ |$ on $\mathbf{F}$ is trivial, then $\mathcal{A}(1)$ is simply the formal power series ring $\mathbf{F}[[z]]$ and $\mathcal{A}[1]$ is the polynomial ring $\mathbf{F}[z]$. The ring of analytic functions on the annulus $A[1, 1]$ are simply elements of $\mathbf{F}[z, z^{-1}]$.

1.3 The Schnirelman Integral and an Analog of the Cauchy Integral Formula

If you think back to your first course on complex function theory, probably nothing stands out as much as the Cauchy Integral Theorem and the Cauchy Integral Formula. Thus, I feel it is most appropriate to begin with a discussion of an integral introduced by Schnirelman [Schn] that serves as an analog to the path integral around a circle so commonplace in complex analysis and from which analogs of many of the usual first theorems in complex analysis can be derived. I point out, however, that the Schnirelman integral is not used much anymore, and the consequences of this integral that I will explain in this section can also be derived by the techniques to be introduced in future lectures. My lectures here on the Schnirelman integral are based on [Adms], but I have changed the definition to make a closer parallel with classical complex function theory.

Definition and Basic Properties

Consider the homomorphism from $\mathbf{Z}$ to $\mathbf{F}$ defined by sending an integer n to $n \cdot 1$ in $\mathbf{F}$. If $\mathbf{F}$ has characteristic zero, this homomorphism is injective, and otherwise its image is the prime field of $\mathbf{F}$. When we write $|n|$ in this section, by abuse of notation, we will mean the absolute value of the image of n in $\mathbf{F}$, even when $\mathbf{F}$ has positive characteristic, so for instance $|n| = 0$ if n is divisible by the characteristic of $\mathbf{F}$.

Exercise 1.3.1. *The set of n in $\mathbf{Z}$ such that $|n| < 1$ forms a prime ideal of $\mathbf{Z}$.*

As a consequence of Exercise 1.3.1, there are infinitely many positive integers n such that $|n| = 1$.

Definition. *For an integer $n \geq 1$ such that $|n| = 1$, denote by $\xi_1^{(n)}, \ldots, \xi_n^{(n)}$ the n n-th roots of unity in $\mathbf{F}$. Given a and r in $\mathbf{F}$ and given a function f such that f is defined at all points of the form $a + r\xi_k^{(n)}$ for all $n \geq 1$ with $|n| = 1$ and all $1 \leq k \leq n$, define*

$$\int\limits_{|z-a|=|r|} f(z)dz = \lim_{\substack{n\to\infty \\ |n|=1}} \frac{r}{n} \sum_{k=1}^{n} f\left(a + r\xi_k^{(n)}\right) \xi_k^{(n)},$$

provided the limit on the right exists.

The integral in the above definition is called the **Schnirelman integral,** and if it exists, the function f is called **Schnirelman integrable** on the discrete circle $|z-a| = |r|$. It is clear from the definition that the Schnirelman integral satisfies the usual linearity properties we expect from an integral.

Caution! The above definition for the Schnirelman integral is non-standard. Also, it could be that

$$\int\limits_{|z-a|=|r_1|} f(z)dz \neq \int\limits_{|z-a|=|r_2|} f(z)dz,$$

or even that one of the above integrals exists and the other does not, even if $|r_1| = |r_2|$. This is not well-reflected in my choice of notation.

Proposition 1.3.2. *If $\int\limits_{|z-a|=|r|} f(z)dz$ exists, then*

$$\left| \int\limits_{|z-a|=|r|} f(z)dz \right| \leq |r| \max_{|z-a|=|r|} |f(z)|,$$

provided the right hand side is well-defined.

Proof. Trivial, noting that $|\xi_k^{(n)}| = 1$. □

Proposition 1.3.3. *If $\sum f_j$ converges uniformly on $|z-a| = |r|$ to f and if each f_j is Schnirelman integrable on $|z-a| = |r|$, then f is Schnirelman integrable on $|z-a| = |r|$ and*

$$\int_{|z-a|=|r|} f(z)dz = \sum_j \int_{|z-a|=|r|} f_j(z)dz.$$

Proof. Let $\varepsilon > 0$. By the hypothesis of uniform convergence of the sum, we have

$$\left| f(z) - \sum_{j=0}^{J} f_j(z) \right| < \frac{\varepsilon}{3|r|}$$

for all sufficiently large J and for all z such that $|z-a| = |r|$. Hence, for any n such that $|n| = 1$,

$$\left| \frac{r}{n} \sum_{k=1}^{n} f\left(a + r\xi_k^{(n)}\right) \xi_k^{(n)} \;-\; \frac{r}{n} \sum_{k=1}^{n} \sum_{j=0}^{J} f_j\left(a + r\xi_k^{(n)}\right) \xi_k^{(n)} \right| < \frac{\varepsilon}{3},$$

for all sufficiently large J. Since $|f_j|$ tends uniformly to zero on $|z-a| = |r|$, we have by Proposition 1.3.2, that

$$\left| \sum_{j=0}^{J} \int_{|z-a|=|r|} f_j(z)dz \;-\; \sum_{j=0}^{\infty} \int_{|z-a|=|r|} f_j(z)dz \right| < \frac{\varepsilon}{3},$$

also for all sufficiently large J. Fix J sufficiently large that the above inequalities hold. By the integrability of the f_j, there exists an N such that if $n \geq N$ and $|n| = 1$, then

$$\left| \frac{r}{n} \sum_{k=1}^{n} \sum_{j=0}^{J} f_j\left(a + r\xi_k^{(n)}\right) \xi_k^{(n)} \;-\; \sum_{j=0}^{J} \int_{|z-a|=|r|} f_j(z)dz \right| < \frac{\varepsilon}{3}.$$

Hence, if $n \geq N$ and $|n| = 1$, then we have

$$\left| \frac{r}{n} \sum_{k=1}^{n} f\left(a + r\xi_k^{(n)}\right) \xi_k^{(n)} \;-\; \sum_{j=0}^{\infty} \int_{|z-a|=|r|} f_j(z)dz \right| < \varepsilon. \quad \square$$

Cauchy Integral Theorem and Cauchy Integral Formula

Lemma 1.3.4. *Let $1 \leq |j| < n$ be integers [here $|j|$ denotes the usual Archimedean absolute value of the index j]. Then*

$$\sum_{k=1}^{n} \left(\xi_k^{(n)}\right)^j = 0.$$

Proof. Since $\{\xi_1^{(n)}, \ldots, \xi_n^{(n)}\} = \{(\xi_1^{(n)})^{-1}, \ldots, (\xi_n^{(n)})^{-1}\}$, it suffices to consider j positive. Let $x_1, \ldots, x_n$ be variables. Then, $x_1^j + \cdots + x_n^j$ is a polynomial in the elementary symmetric functions $\sigma_1(x_1, \ldots, x_n), \ldots, \sigma_j(x_1, \ldots, x_n)$ with no constant term. Since $\sigma_i(\xi_1^{(n)}, \ldots, \xi_n^{(n)}) = 0$ for $1 \leq i < n$, the lemma follows. □

Theorem 1.3.5 (Cauchy Integral Theorem). *Let $\mathbf{B}_{\leq R}(a) = \{z \in \mathbf{F} : |z-a| \leq R\}$ denote the closed ball of radius R centered at a. Let f be analytic on $\mathbf{B}_{\leq R}(a)$. Let $r \in \mathbf{F}$ with $|r| = R$. Then, f is Schnirelman integrable on $|z-a| = |r|$ and*

$$\int_{|z-a|=|r|} f(z)dz = 0.$$

Proof. Without loss of generality, we may assume $a = 0$. By linearity and Proposition 1.3.3, it suffices to show the theorem for $f(z) = z^j$ for $j \geq 0$. The theorem then follows from Lemma 1.3.4 since the expression inside the limit defining the Schnirelman integral vanishes as soon as $n \geq j + 2$. $\square$

Given a formal power series $f(z) = \sum a_j z^j$, define the k-th Hasse derivative of f by

$$D^k f(z) = \sum_{j=k}^{\infty} a_j \binom{j}{k} z^{j-k}.$$

Observe that in characteristic zero, the Hasse derivative $D^k f$ is simply $f^{(k)}/k!$. Hasse derivatives are more useful than ordinary derivatives in positive characteristic and have similar properties.

Theorem 1.3.6 (Cauchy Integral Formula). *Let f be analytic on $\mathbf{B}_{\leq R}(a)$, let $r \in \mathbf{F}$ with $|r| = R$, let $w \in \mathbf{F}$ with $|w - a| \neq R$, and let $n \geq 0$. Then*

$$\int\limits_{|z-a|=|r|} \frac{f(z)}{(z-w)^{(n+1)}}\, dz = \begin{cases} D^n f(w) & \text{if } |w-a| < R \\ 0 & \text{if } |w-a| > R. \end{cases}$$

Proof. From the definition and from Lemma 1.3.4, we see that if k is an integer, then

$$\int\limits_{|z|=|r|} z^k dz = \begin{cases} 1 & \text{if } k = -1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Without loss of generality, we consider $a = 0$, and write $f(z)$ as a power series,

$$f(z) = \sum_{j=0}^{\infty} a_j z^j.$$

If $w = 0$, then the theorem follows from (1) and Proposition 1.3.3. If $0 < |w| < R$, then

$$\sum_{k=n}^{\infty} \binom{k}{n} \left(\frac{w}{z}\right)^{k-n}$$

converges uniformly to $1/(1 - w/z)^{n+1}$ on $|z| = R$. Hence,

$$\int\limits_{|z|=|r|} \frac{f(z)}{(z-w)^{n+1}} dz = \sum_{k=n}^{\infty}\sum_{j=0}^{\infty} w^{k-n} \int\limits_{|z|=|r|} a_j \binom{k}{n} z^{j-k-1} dz = \sum_{k=n}^{\infty} a_k w^{k-n} \binom{k}{n} = D^n f(w),$$

where the second to last equality follows from (1). If $|w| > R$, we use a similar argument with

$$\frac{1}{(z-w)^{n+1}} = \frac{(-1)^{n+1}}{w^{n+1}} \sum_{k=n}^{\infty} \binom{k}{n} \left(\frac{z}{w}\right)^k$$

to conclude that the integral is zero, since there will be no negative powers of z. $\square$

Theorem 1.3.7 (Residue Theorem). *Let a and r be elements of $\mathbf{F}$. Let $f(z)$ be analytic on $|z-a| \leq |r|$, let $P(z)$ be a polynomial with no zeros on $|z-a| = |r|$, and let $R(z) = f(z)/P(z)$. Then,*

$$\int\limits_{|z-a|=|r|} R(z) dz = \sum_{|b-a|<r} \mathrm{Res}(R, b).$$

Proof. Take the partial fraction expansion of R,

$$R(z) = g(z) + \frac{A_{1,1}}{z-b_1} + \cdots + \frac{A_{1,m_1}}{(z-b_1)^{m_1}} + \cdots + \frac{A_{n,1}}{z-b_n} + \cdots + \frac{A_{n,m_n}}{(z-b_n)^{m_n}},$$

where g is analytic on $|z-a| \leq |r|$ and apply Theorem 1.3.6. $\square$

1.4 Consequences of the Cauchy Integral Formula

Maximum Modulus Principle

Theorem 1.4.1 (Maximum Modulus Principle). *Let r and a be in $\mathbf{F}$, and let f be analytic on $|z-a| \le |r|$. Then,*

$$\max_{|z-a|\le|r|} |f(z)| = \max_{|z-a|=|r|} |f(z)|.$$

Proof. Let w be in $\mathbf{F}$ such that $|w-a| < |r|$. Then, by Theorem 1.3.6 and Proposition 1.3.2,

$$|f(w)| = \left| \int\limits_{|z-a|=|r|} \frac{f(z)}{z-w} dz \right| \le |r| \max_{|z-a|=|r|} \frac{|f(z)|}{|z-w|}.$$

Now, $|z-w| = |(z-a)-(w-a)| = |z-a| = |r|$ by Exercise 1.1.1, and hence the theorem follows. □

Remark. In complex analysis, there is a stronger form of the maximum modulus principle. Namely if f is analytic on $|z-a| = r$ and if f attains its maximum in the interior, then f must be constant. This is easily seen to be false in non-Archimedean function theory. Indeed, consider $|c| > 1$ and let $f(z) = z + c$. Then, $|f(z)| = |c|$ for all $|z| \le 1$, yet f is not constant. Notice, however, that $|f|$ is constant on $|z| \le 1$.

A variation on the maximum principle is

Proposition 1.4.2. *Let r and a be in $\mathbf{F}$, and let f be analytic on $|z-a| \le |r|$. Then,*

$$|D^n f(w)| \le \frac{\max\limits_{|z-a|=|r|} |f(z)|}{|r|^n}$$

for all w in $\mathbf{F}$ such that $|w-a| < |r|$ and all integers $n \ge 0$.

Proof. Fix w in $\mathbf{F}$ such that $|w-a| < |r|$. Then, by Theorem 1.3.6 and Proposition 1.3.2, as in the proof of the maximum principle, we have

$$|D^n f(w)| = \left| \int\limits_{|z-a|=|r|} \frac{f(z)}{(z-w)^{n+1}} dz \right| \le |r| \max_{|z-a|=|r|} \frac{|f(z)|}{|z-w|^{n+1}}.$$

Again, $|z-w| = |(z-a)-(w-a)| = |z-a| = |r|$ by Exercise 1.1.1, and hence the proposition follows. □

Proposition 1.4.3. *Let a in $\mathbf{F}$ and let $f(z) = \sum_{j=0}^{\infty} c_j (z-a)^j$ be an analytic function with radius of convergence $0 < R \le \infty$. Let b be an element of $\mathbf{F}$ such that $|b-a| < R$. Then,*

$$f(z) = \sum_{n=0}^{\infty} D^n f(b)(z-b)^n,$$

and this second series has radius of convergence R as well.

Proof. The case of trivial absolute value, in which case both series are polynomials or formal power series, is clear. Hence, we assume the absolute value on $\mathbf{F}$ is non-trivial. Fix z in $\mathbf{F}$ such that $|z-b| < R$. Then $|z-a| = |(z-b)-(b-a)| < R$, and we can find r in $\mathbf{F}$ such that

$$\max\{|z-b|, |b-a|\} < |r| < R.$$

By Proposition 1.4.2,

$$|D^n f(b)(z-b)^n| \le M\left|\frac{z-b}{r}\right|^n, \qquad \text{where } M = \max_{|z-a|=|r|}|f(z)|.$$

As $|(z-b)/r|<1,$ we see the series

$$\sum_{n=0}^{\infty} D^n f(b)(z-b)^n$$

converges at z. Hence the radius of convergence of this series is at least R. By symmetry, it is at most R.

Once we have convergence, we easily see

$$\begin{aligned}\sum_{n=0}^{\infty} D^n f(b)(z-b)^n &= \sum_{n=0}^{\infty}\sum_{j=n}^{\infty} c_j\binom{j}{n}(b-a)^{j-n}(z-b)^n\\ &= \sum_{j=0}^{\infty} c_j\sum_{n=0}^{j}\binom{j}{n}(b-a)^{j-n}(z-b)^n\\ &= \sum_{j=0}^{\infty} c_j(z-b+b-a)^j = \sum_{j=0}^{\infty} c_j(z-a)^j = f(z).\end{aligned}$$ □

Identity Theorem

Theorem 1.4.4 (Identity Theorem). *Let f be analytic on $\mathbf{B}_{\le R}(a)$ and let $z_1, z_2, z_3, \ldots$ be points in $\mathbf{B}_{\le R}(a)$ such that z_0 is an accumulation point of the z_k in $\mathbf{B}_{\le R}(a)$. If $f(z_k)=0$ for all $k\ge 1$, then $f\equiv 0$.*

Proof. By Proposition 1.4.3, we can expand f as a power series about z_0,

$$f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j,$$

and this series will also have radius of convergence R. If not all the a_j are zero, let j_0 be the smallest index such that $a_{j_0}\ne 0$. Then, $f(z)/(z-z_0)^{j_0}$ is continuous and non-zero for $|z-z_0|$ small. This contradicts the hypotheses that $f(z_k)=0$ and z_k accumulates at z_0. □

Liouville's Theorem

Theorem 1.4.5 (Liouville's Theorem). *A bounded entire function must be constant. Moreover, if*

$$\frac{|f(z)|}{|z|^d}$$

remains bounded as $|z|\to\infty$, then f must be a polynomial of degree at most d.

Proof. Write f as a power series

$$\sum_{j=0}^{\infty} a_j z^j.$$

If f is not a polynomial of degree at most d, then there is some coefficient $a_{j_0}\ne 0$ with $j_0>d$. By Theorem 1.3.6

$$a_{j_0} = \int\limits_{|z|=|r|} \frac{f(z)}{z^{j_0+1}}dz.$$

Then, by Proposition 1.3.2,

$$|a_{j_0}| \leq \frac{\max\limits_{|z|=|r|} |f(z)|}{|r|^{j_0}}.$$

Because $j_0 > d$, the right-hand side tends to zero as $|r| \to \infty$, contradicting $a_{j_0} \neq 0$. $\square$

The Schwarz Lemma

Theorem 1.4.6 (Schwarz Lemma). *Let f be analytic on $\mathbf{B}_{<1}$ such that the image of f is contained in $\mathbf{B}_{\leq 1}$ and such that $f(0) = 0$. Then, $|f(z)| \leq |z|$ for all $|z| < 1$ and $|f'(0)| \leq 1$.*

Proof. If the absolute value on $\mathbf{F}$ is trivial, then so is the statement of the theorem. We therefore assume that the absolute value is non-trivial. Hence, choose a sequence r_n with $|r_n| < 1$ and such that $|r_n| \to 1$. By the assumption that $f(0) = 0$, the function $g(z) = f(z)/z$ is also analytic on $\mathbf{B}_{<1}$ and by the maximum modulus principle, for $|z| < |r_n|$, we have

$$|g(z)| \leq \frac{1}{|r_n|}.$$

Taking the limit as $|r_n| \to 1$ completes the proof of the theorem. $\square$

Remark. In the complex Schwarz Lemma, one has the additional statement that if equality holds at some point, then $f(z) = \alpha z$ for some $|\alpha| = 1$. This is easily seen to be false in the non-Archimedean case by considering, for example, $f(z) = z(1 + z)$.

Corollary 1.4.7 (Schwarz-Pick). *Let f be analytic on $\mathbf{B}_{<1}$ such that the image of f is contained in $\mathbf{B}_{\leq 1}$. Then, for all z and w in $\mathbf{B}_{<1}$, we have $|f(z) - f(w)| \leq |z - w|$.*

Proof. Fix z and w in $\mathbf{B}_{<1}$. Consider $g(\zeta) = f(\zeta + w) - f(w)$. Then, if $|\zeta| < 1$, we have $|\zeta + w| < 1$, and so

$$|g(\zeta)| = |f(\zeta + w) - f(w)| \leq 1,$$

and hence g satisfies the hypotheses of the theorem. Now choosing $\zeta = z - w$ gives the corollary. $\square$

Remark. The complex Schwarz-Pick lemma says that holomorphic self-maps of the unit disc are distance non-increasing in the hyperbolic metric. Corollary 1.4.7 says that analytic self-maps of a non-Archimedean disc are distance non-increasing in the standard non-Archimedean metric.

1.5 Morera's Theorem?

I will conclude my first lecture by pointing out that there is no converse of the Cauchy Integral Theorem for Schnirelman integrals, that is no analog of Morera's Theorem.

Example. Let $\mathbf{F}$ be a complete non-Archimedean field which contains transcendental numbers and consider the function f which is 1 at all algebraic numbers in $\mathbf{F}$ and 0 at all transcendental numbers of $\mathbf{F}$. Clearly f is not given by a power series. On the other hand, let a and $r \neq 0$ be elements of f. Fix a positive integer n with $|n| = 1$. Suppose that two of the numbers $a + r\xi_i^{(n)}$ and $a + r\xi_j^{(n)}$ are algebraic. Then, $r(\xi_i^{(n)} - \xi_j^{(n)})$ is algebraic and non-zero; hence, r is algebraic. Therefore a is also algebraic. Thus, given r and a, one of three things can happen: $a + r\xi_i^{(n)}$ is algebraic for all $i = 1, \ldots, n$; $a + r\xi_i^{(n)}$ is transcendental for all $i = 1, \ldots, n$; or $a + r\xi_i^{(n)}$ is algebraic for exactly one $i = 1, \ldots, n$. In any of these cases, we see that the sum defining the Schnirelman integral of f tends to zero as $n \to \infty$.

1.6 Concluding Remarks

The Schnirelman integral seems never to have been a widely known technique and is not often used in non-Archimedean function theory. The methods I will discuss in the next lecture can be used to prove the results of this section, and in fact some stronger results. However, the Schnirelman integral is a useful construct to have in one's bag of tricks because, in particular, it often allows one to convert standard complex analytic proofs to the non-Archimedean setting. For example, in the 1960's, Adams [Adms] made extensive use of the Schnirelman integral to prove p-adic versions of the Gelfond-Schneider-Lang transcendence machinery, and many of his proofs follow the same general outline of their complex counterparts.

2 Valuation Polygons and a Poisson-Jensen Formula

In the previous lecture, Schnirelman integrals were introduced so that non-Archimedean analogs of familiar results from classical complex function theory could be proved in a manner reminiscent of the proofs most familiar from an introductory course in complex analysis. Although the viewpoint of the previous lecture is nice in that it emphasizes similarity between complex and non-Archimedean function theory, in fact, there are many differences between non-Archimedean function theory and its classical complex counterpart. What has been, in practice, a more useful tool than the Schnirelman integral is a set of techniques for determining the locations of non-Archimedean zeros of power series that goes by either the name of the the valuation polygon or the Newton polygon. This is a powerful technique available in non-Archimedean analysis that has no exact parallel in complex analysis. Mastering these techniques is essential for developing non-Archimedean analysis, and these techniques can generally be used in place of the Cauchy Integral Formula and often give stronger results. Takashi Harase, in his review [Har] of one of Dwork's last papers [Dwk], writes:

> "The author uses the Newton polygon at the places where classical analysts use Cauchy's integral formula. He could be called the magician of the Newton polygon."

Much of the text of this lecture is taken from [ChWa].

2.1 The Residue Class Field

Non-Archimedean fields have some structure not available in the complex numbers or other Archimedean fields. Observe that the set

$$\mathcal{O} = \{z \in \mathbf{F} : |z| \le 1\}$$

forms a subring of $\mathbf{F}$. The subring $\mathcal{O}$ is called the **ring of integers** of $\mathbf{F}$. Denote by

$$M = \{z \in \mathbf{F} : |z| < 1\}.$$

Exercise 2.1.1. *Show that M is the unique maximal ideal of $\mathcal{O}$.*

A ring with a unique maximal idea is called a **local ring**. We denote by $\widetilde{\mathbf{F}}$ the field $\mathcal{O}/M$, which is called the **residue class field of F**. Given an element a in $\mathcal{O}$, we denote by $\tilde{a}$ its image in $\widetilde{\mathbf{F}}$.

Exercise 2.1.2. *If $\mathbf{F}$ is algebraically closed, then $\widetilde{\mathbf{F}}$ is too.*

2.2 Non-Archimedean Absolute Values on Rings of Analytic Functions

Let $r_1 \le r_2$ and consider the ring $\mathcal{A}[r_1, r_2]$ of analytic functions on the bordered annulus $A[r_1, r_2]$. The elements of $\mathcal{A}[r_1, r_2]$ are Laurent series of the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \text{ such that } \lim_{|n|\to\infty} |c_n| r^n = 0 \text{ for all } r_1 \le r \le r_2.$$

Remark. Unlike in the previous lecture, here $|n|$ for an integer index n refers to the usual Archimedean absolute value of n, and not the absolute value of the image of n in $\mathbf{F}$. Also, in the previous lecture we tended to use r to denote an element of $\mathbf{F}$. In this section, r will denote a non-negative real number.

For each r between r_1 and r_2, we define

$$\boxed{|f|_r = \sup |c_n| r^n.}$$

Remark. Observe that for fixed f, we easily see that $|\ |_r$ is a continuous function of r. If $r_1 = 0$, so that no negative powers appear in the series expansion for f, we also see that $|f|_r$ is non-decreasing in r.

Next, we highlight one technical point.

Remark. Recall that we use $|\mathbf{F}^\times|$ to denote the value group

$$|\mathbf{F}^\times| = \{|a| \in \mathbf{R}_{>0} : a \in \mathbf{F}\}.$$

It could be that $|\mathbf{F}^\times| \neq \mathbf{R}_{>0}$, and this often creates some technicalities in non-Archimedean proofs where one has to consider separately the cases $r \in |\mathbf{F}^\times|$ and $r \notin |\mathbf{F}^\times|$.

We will see in Proposition 2.2.3 below that $|\ |_r$ is in fact a non-Archimedean absolute value if $r > 0$, but first we state another version of the non-Archimedean maximum modulus principle.

Proposition 2.2.1 (Maximum Modulus Principle). *If f is analytic on $A[r_1, r_2]$ and z_0 is in $A[r_1, r_2]$, then*

$$|f(z_0)| \le |f|_{|z_0|}.$$

Moreover, if $|z| = |z_0|$ and $|f(z)| < |f|_{|z_0|}$, then $\widetilde{\frac{z}{z_0}}$ is one of at most finitely many residue classes in $\widetilde{\mathbf{F}}$.

Proof. That $|f(z_0)| \le |f|_{|z_0|}$ follows immediately from the non-Archimedean triangle inequality, so we need to show that equality holds outside at most finitely many residue classes. Write

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n,$$

and let c be an element of $\mathbf{F}$ such that $|c| = |f|_{|z_0|}$. Let

$$g(z) = \sum_{n=-\infty}^{\infty} b_n z^n, \qquad \text{where } b_n = \frac{c_n z_0^n}{c}.$$

Note that $\sup |b_n| = 1$, and in particular g has coefficients in $\mathcal{O}$. Let

$$\tilde{g}(z) = \sum_{n=-\infty}^{\infty} \tilde{b}_n z^n,$$

and note that $\tilde{g}$ is not identically zero. Then, directly from the definitions, if $|z| = |z_0|$ and $|f(z)| < |f|_{|z_0|}$, then

$$\tilde{g}\left(\widetilde{\frac{z}{z_0}}\right) = 0.$$

Because

$$\lim_{|n|\to\infty} |b_n| = 0,$$

$\tilde{g}$ has only finitely many non-zero coefficients, and hence, there are only finitely many possibilities for $\widetilde{\frac{z}{z_0}}$. $\square$

Corollary 2.2.2. *If f is analytic on $A[r_1, r_2]$ and the set $\{|f|_r : r_1 \le r \le r_2\}$ is bounded in* $\mathbf{R}$, *then f is bounded on $A[r_1, r_2]$.*

Proposition 2.2.3. *If f and g are analytic functions on $A[r_1, r_2]$, then*

$$\begin{aligned} |f+g|_r &\le \max\{|f|_r, |g|_r\} && \text{[non-Archimedean triangle inequality]} \\ |fg|_r &= |f|_r |g|_r && \text{[multiplicativity]} \end{aligned}$$

for $r_1 \le r \le r_2$.

Proof. The triangle inequality is clear. If $r \in |\mathbf{F}^\times|$, then multiplicativity follows from Proposition 2.2.1. For $r \notin |\mathbf{F}^\times|$, the case when $|\ |$ is trivial is left as an exercise for the reader. If $|\ |$ is non-trivial, we can find a sequence $r_n \in |\mathbf{F}^\times|$ such that $r_n \to r$, and then the proposition follows from the continuity of $|\ |_r$ in r. $\square$

Proposition 2.2.3 says that each $|\ |_r$ is a non-Archimedean absolute value on the ring of functions analytic on an annulus containing $\{z : |z| = r\}$, provided $r > 0$.

Proposition 2.2.4. *The ring of analytic functions on $A[r_1, r_2]$ is complete with respect to the norm*

$$|f|_{\sup} = \sup_{r_1 \le r \le r_2} |f|_r.$$

Proof. Let

$$f_n(z) = \sum_m a_{n,m} z^m$$

be a Cauchy sequence. Then, for n and n' sufficiently large, we have

$$\varepsilon > |f_n - f_{n'}|_{\sup} = \sup_r \sup_m |a_{n,m} - a_{n',m}| r^m. \tag{2}$$

This implies that for each fixed m, the sequence of coefficients $a_{n,m}$ is Cauchy, and hence converges to some b_m. Let

$$f(z) = \sum_m b_m z^m \in \mathbf{F}[[z, z^{-1}]].$$

First, we need to check that $|f|_r < \infty$, for $r_1 \le r \le r_2$. Since $a_{n,m} \to b_m$, if $b_m \ne 0$, then $|a_{n,m}| = |b_m|$ for n sufficiently large. Thus,

$$|b_m| r^m = |a_{n,m}| r^m \le |f_n|_r \le |f_n|_{\sup} \le \sup_n |f_n|_{\sup} < \infty.$$

The last inequality follows from the assumption that f_n is Cauchy. Second, we need to check that $|f_n - f|_{\sup} \to 0$. Because $a_{n,m} \to b_m$, for n_m sufficiently large possibly depending on m, we have

$$\sup_r |b_m - a_{n_m,m}| r^m < \varepsilon.$$

On the other hand, since the f_n are Cauchy, for n' sufficiently large and independent of m, inequality (2) is satisfied. Therefore,

$$\sup_r \sup_m |b_m - a_{n',m}| r^m < \sup_r \sup_m \max\{|b_m - a_{n_m,m}|, |a_{n_m,m} - a_{n',m}|\} r^m < \varepsilon. \quad \square$$

By multiplicativity in Proposition 2.2.3, we can extend $|\ |_r$ to meromorphic functions. We will need that meromorphic functions are analytic away from poles.

Proposition 2.2.5. *Let f be analytic on $\mathbf{B}_{\le r_1}$ with $r_1 > 0$ and $f(0) \ne 0$. Let*

$$r_2 = \sup\{r \le r_1 : |f|_r = |f(0)|\} > 0.$$

Then, there exists a unique analytic function g on $\mathbf{B}_{<r_2}$ such that $fg = 1$ on $\mathbf{B}_{<r_2}$.

Proof. The uniqueness of g is clear. By the non-Archimedean triangle inequality and the choice of r_2, we have

$$\left|1 - \frac{f}{f(0)}\right|_r < 1, \qquad \text{for all } r < r_2.$$

Hence,

$$\sum_{j=0}^{\infty} \left(1 - \frac{f}{f(0)}\right)^j$$

converges to an analytic function h on $\mathbf{B}_{<r_2}$ by Proposition 2.2.4. Here we are also using that a sequence of functions converges in $\mathcal{A}(r_2)$ if and only if that sequence converges in $\mathcal{A}[r]$ for all $r < r_2$. Finally, set $g = h/f(0)$. □

Liouville's Theorem Again and the Riemann Extension Theorem

We now see how non-Archimedean analogs of Liouville's Theorem and the Riemann Extension Theorem follow easily from the basic properties of $|\ |_r$. The proof of each proposition is similar, so we state both propositions first and then give a joint proof.

Proposition 2.2.6 (Liouville's Theorem). *If f is entire and $|f|_r$ is bounded for all r, then f is constant.*

We say that an analytic function f on $A[r_1, \infty)$ is **analytic at infinity** if $f(1/z)$ is an analytic function on $A[0, 1/r_1] = \mathbf{B}_{\le r_1^{-1}}$.

Proposition 2.2.7 (Riemann Extension Theorem). *If f is analytic on $A[r_1, \infty)$ and the set $\{|f|_r : r \ge r_1\}$ is bounded in $\mathbf{R}$, then f is analytic at infinity.*

Proof of Propositions 2.2.6 and 2.2.7. Write $f(z) = \sum c_n z^n$. To prove either proposition, we need to prove that the existence of an index $n_0 > 0$ such that $c_{n_0} \neq 0$ contradicts the boundedness of $|f|_r$. But, if such a n_0 exits, then

$$|f|_r \ge |c_{n_0}| r^{n_0} \to \infty \text{ as } r \to \infty.$$ □

An analytic function f on $A[r_1, \infty)$ is said to be **meromorphic at infinity** if $z^{-m} f(z)$ is analytic at infinity for some integer $m \ge 0$. If an analytic function f on $A[r_1, \infty)$ is not meromorphic at ∞, then it is said to have **an essential singularity at infinity.** We now state a proposition that says if $|f|_r$ grows slowly as $r \to \infty$, then it cannot have an essential singularity at infinity.

Proposition 2.2.8. *If f is analytic on $A[r_1, \infty)$ and*

$$\limsup_{r \to \infty} \frac{\log |f|_r}{\log r} < \infty,$$

then f is meromorphic at infinity.

Proof. Let $g(z) = z^{-m} f(z)$, which is also an analytic function on $A[r_1, \infty)$. Choose m larger than

$$\limsup_{r \to \infty} \frac{\log |f|_r}{\log r}.$$

Then,

$$\limsup_{r \to \infty} \log |g|_r < \infty.$$

Hence g is analytic at infinity by Proposition 2.2.7. □

2.3 Valuation Polygons

Preliminaries

Let f be analytic on $A[r_1, r_2]$. Write

$$f(z) = \sum_{n \in \mathbf{Z}} c_n z^n.$$

For r with $r_1 \leq r \leq r_2$, let

$$\boxed{k(f,r) = \inf\{n \in \mathbf{Z} : |c_n| r^n = |f|_r\} \quad \text{and} \quad K(f,r) = \sup\{n \in \mathbf{Z} : |c_n| r^n = |f|_r\}.}$$

Note that in the special case of $r = 0$ with $f(0) = 0$, we define

$$k(f,0) = 0 \qquad \text{and} \qquad K(f,0) = \inf\{n : c_n \neq 0\}.$$

The integer $K(f,r)$ is often call the **central index** and sometimes also has a role in complex analysis.

A radius r such that $K(f,r) > k(f,r)$ is called a **critical radius**.

Proposition 2.3.1. *The set of critical radii for a Laurent series is discrete.*

Proof. Let f be a Laurent series with critical radius r', so $K(f,r') > k(f,r')$. Let $K = K(f,r')$ and $k = k(f,r')$. If $n > k$, then either $a_n = 0$ or $|a_n|(r')^n \leq |a_k|(r')^k$. Hence, if $r < r'$, then

$$|a_n| r^n = \left(\frac{r}{r'}\right)^n |a_n|(r')^n \leq \left(\frac{r}{r'}\right)^n |a_k|(r')^k = \left(\frac{r}{r'}\right)^{n-k} |a_k| r^k < |a_k| r^k,$$

and so $K(f,r) \leq k(f,r')$. Let m be the largest integer $< k$ such that $a_m \neq 0$. If no such integer m exists, then $K(f,r) = k(f,r) = k$ for all $r < r'$, and so there are no critical radii smaller than r'. Otherwise, let r'' be the radius such that $|a_m|(r'')^m = |a_k|(r'')^k$. Because, $|a_m|(r')^m < |a_k|(r')^k$, we know $r'' < r'$. Hence, for $r'' < r < r'$, we have $K(f,r) = k(f,r)$, and so there are no critical radii between r'' and r'. By a similar argument, we see that $k(f,r) \geq K(f,r')$ for $r \geq r'$, and if M is the smallest integer greater than K such that $a_M \neq 0$, then $K(f,r) = k(f,r) = K$ for all $r' < r < r'''$, where $r''' > r'$ is such that $|a_K|(r''')^K = |a_M|(r''')^M$. □

Proposition 2.3.2. *If f and g are analytic on $A[r,r]$ then*

$$K(fg,r) = K(f,r) + K(g,r) \qquad \textit{and} \qquad k(fg,r) = k(f,r) + k(g,r).$$

Proof. We provide the proof for K. The proof for k is similar, or follows by changing z to z^{-1}. Write $f(z) = \sum a_n z^n$, $g(z) = \sum b_n z^n$, and $fg(z) = \sum c_n z^n$. Let $m = K(f,r) + K(g,r)$. Then,

$$c_m = \sum_{i+j=m} a_i b_j.$$

One of the terms in this sum comes from $i = K(f,r)$ and $j = K(g,r)$. In this case

$$|a_i b_j| = \frac{|fg|_r}{r^m}.$$

If $i < K(f,r)$, then $j > K(g,r)$, and so $|b_j| < |g|_r / r^j$. But, $|a_i| \leq |f|_r / r^i$, and hence

$$|a_i b_j| < \frac{|fg|_r}{r^m}.$$

Similarly, $|a_i b_j| < |fg|_r / r^m$ if $i > K(f,r)$. Hence, $|c_m| = |fg|_r / r^m$. This shows that

$$K(fg,r) \geq K(f,r) + K(g,r).$$

On the other hand, if $i + j > m$, then either $i > K(f,r)$ or $j > K(g,r)$, and so we see $K(fg,r) \leq K(f,r) + K(g,r)$. □

Proposition 2.3.3. *Let $r > 0$. If f is analytic on $A[r,r]$ with $K(f,r) = k(f,r)$, then f is invertible in $A[r,r]$.*

Proof. Note $K(fz^m, r) = K(f,r) + m$, and similarly for k. Thus, by multiplying f by $z^{-k(f,r)}$, a unit in $A[r,r]$, we may assume $K(f,r) = k(f,r) = 0$.

Then, if c_0 is the constant term in the Laurent series defining f, then the assumption that $K(f,r) = k(f,r) = 0$ implies

$$|f - c_0|_r < |c_0| \quad \text{or in other words} \quad |c_0^{-1}f - 1|_r < 1.$$

Thus,

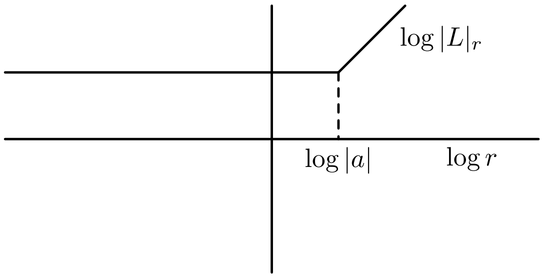
$$f^{-1} = c_0^{-1}[1 - (1 - c_0^{-1}f)]^{-1} = c_0^{-1}[1 + (1 - c_0^{-1}f)^2 + (1 - c_0^{-1}f)^3 + \dots] \qquad \square$$

Valuation Polygons for Polynomials

Consider a monic linear polynomial $L(z) = z - a$. Then, clearly

$$|L|_r = \begin{cases} |a| & \text{if } r \le |a| \\ r & \text{if } r \ge |a|. \end{cases}$$

The following figure is a graph of $\log|L|_r$ as a function of $\log r$.

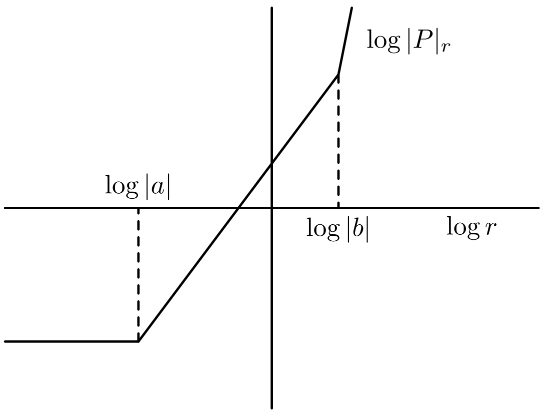


Notice that the corner of the graph indicates that $L(z)$ has a zero with $|z| = |a|$.

Now suppose that $P(z) = (z-a)^n(z-b)^m$ with $0 < |a| < |b|$. Then,

$$\log|P|_r = \begin{cases} n\log|a| + m\log|b| & \text{if } r \le |a| \\ n\log r + m\log|b| & \text{if } |a| \le r \le |b| \\ (n+m)\log r & \text{if } r \ge |b|. \end{cases}$$

This time the graph of $\log|P|_r$ as a function of $\log r$ looks like



Again, we see a piecewise linear graph whose corners indicate the location of the zeros of P. Notice also that the change in slope indicates the number of zeros with that absolute value: in this case the slope goes from 0 to n at $r = |a|$ and from n to $n+m$ at $r = |b|$. The above graph is called the **valuation polygon** of P. Observe that for the above example, $K(P,|a|) = n$ and $k(P,|a|) = 0$, that $K(P,|b|) = m+n$ and $k(P,|b|) = n$, and that $K(P,r) = k(P,r)$ for all $r \neq |a|,|b|$. Thus, the corners of the valuation polygon correspond to the critical radii.

If P is an arbitrary polynomial, we can write

$$P(z) = cz^{m_0}\prod_j (z-a_j)^{m_j},$$

and we see that $\log r \mapsto \log|P|_r$ is a piecewise linear function whose corners indicate the locations of the zeros at P, and such that the change in slope at the corners indicates the number of zeros, counting multiplicity, that P has at that absolute value.

Remark. What is known as the **Newton polygon** is a polygon dual to the valuation polygon in a certain sense. As such, the Newton polygon of a polynomial P also encodes the locations of the zeros of P, but I prefer the valuation polygon to the Newton polygon for non-Archimedean function theory. See [Rob, pp. 300] for a more detailed description of the Newton polygon and its relationship to the valuation polygon.

We now show that for a polynomial P, the corners of the valuation polygon occur precisely at the critical radii and that P has $K(P,r)-k(P,r)$ zeros, counting multiplicity, with absolute value r.

Proposition 2.3.4. *A polynomial P has $K(P,r)-k(P,r)$ zeros, counting multiplicity, of absolute value r.*

Proof. Without loss of generality, we may assume P is monic, and we proceed by induction. The proof essentially amounts to Gauss's Lemma.

The proposition is clear if P is a linear polynomial. Now consider

$$P(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_0 = (z-z_1)(z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0).$$

Let

$$k_1 = \inf\{m : |a_m|r^m = \sup_\ell |a_\ell|r^\ell\} \qquad \text{and} \qquad K_1 = \sup\{m : |a_m|r^m = \sup_\ell |a_\ell|r^\ell\}.$$

By the induction hypotheses, $z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0$ has $K_1 - k_1$ zeros on $|z| = r$.

Now, $|b_0| = |a_0||z_1|$. Hence,

$$|b_0| < \frac{|z_1|}{r}|a_{K_1}|r^{K_1+1} \qquad \text{if } k_1 > 0, \tag{3}$$

and

$$|b_0| = \frac{|z_1|}{r}|a_{K_1}|r^{K_1+1} \qquad \text{if } k_1 = 0. \tag{4}$$

For $0 < j < k_1$ and for $K_1 + 1 < j \leq n$,

$$\begin{aligned} |b_j|r^j &= |a_{j-1} - z_1 a_j|r^j \\ &\leq \max\left\{|a_{j-1}|r^{j-1}, \frac{|z_1|}{r}|a_j|r^j\right\} r \\ &< \max\left\{1, \frac{|z_1|}{r}\right\}|a_{K_1}|r^{K_1+1}. \end{aligned} \tag{5}$$

If $j = k_1 > 0$ and $|z_1| < r$, then

$$\begin{aligned} |b_{k_1}|r^{k_1} &= |a_{k_1-1} - z_1 a_{k_1}|r^{k_1} \\ &\leq \max\left\{|a_{k_1-1}|r^{k_1-1}, \frac{|z_1|}{r}|a_{k_1}|r^{k_1}\right\} r \\ &< |a_{K_1}|r^{K_1+1}. \end{aligned} \tag{6}$$

Similarly, if $j = k_1 > 0$ and $|z_1| \geq r$, then

$$\begin{aligned} |b_{k_1}|r^{k_1} &= |a_{k_1-1} - z_1 a_{k_1}|r^{k_1} \\ &= \max\left\{|a_{k_1-1}|r^{k_1-1}, \frac{|z_1|}{r}|a_{k_1}|r^{k_1}\right\} r \\ &= \frac{|z_1|}{r}|a_{K_1}|r^{K_1+1}. \end{aligned} \tag{7}$$

For $k_1 + 1 \leq j \leq K_1$,

$$\begin{aligned} |b_j|r^j &= |a_{j-1} - z_1 a_j|r^j \\ &\leq \max\left\{|a_{j-1}|r^{j-1}, \frac{|z_1|}{r}|a_j|r^j\right\} r \\ &\leq \max\left\{1, \frac{|z_1|}{r}\right\} |a_{K_1}|r^{K_1+1}. \end{aligned} \tag{8}$$

For $j = k_1 + 1$ and $|z_1| < r$, then

$$\begin{aligned} |b_{k_1+1}|r^{k_1+1} &= |a_{k_1} - z_1 a_{k_1+1}|r^{k_1+1} \\ &= \max\left\{|a_{k_1}|r^{k_1}, \frac{|z_1|}{r}|a_{k_1+1}|r^{k_1+1}\right\} r \\ &= |a_{K_1}|r^{K_1+1}. \end{aligned} \tag{9}$$

For $j = K_1$ and $|z_1| > r$, then

$$\begin{aligned} |b_{K_1}|r^{K_1} &= |a_{K_1-1} - z_1 a_{K_1}|r^{K_1} \\ &= \max\left\{|a_{K_1-1}|r^{K_1-1}, \frac{|z_1|}{r}|a_{K_1}|r^{K_1}\right\} r \\ &= \frac{|z_1|}{r}|a_{K_1}|r^{K_1+1}. \end{aligned} \tag{10}$$

For $j = K_1 + 1 < n$ and $|z_1| > r$, then

$$\begin{aligned} |b_{K_1+1}|r^{K_1+1} &= |a_{K_1} - z_1 a_{K_1+1}|r^{K_1+1} \\ &\leq \max\left\{|a_{K_1}|r^{K_1}, \frac{|z_1|}{r}|a_{K_1+1}|r^{K_1+1}\right\} r \\ &< \frac{|z_1|}{r}|a_{K_1}|r^{K_1+1}. \end{aligned} \tag{11}$$

Similarly, for $j = K_1 + 1 < n$ and $|z_1| \leq r$,

$$\begin{aligned} |b_{K_1+1}|r^{K_1+1} &= |a_{K_1} - z_1 a_{K_1+1}|r^{K_1+1} \\ &= \max\left\{|a_{K_1}|r^{K_1}, \frac{|z_1|}{r}|a_{K_1+1}|r^{K_1+1}\right\} r \\ &= |a_{K_1}|r^{K_1+1}. \end{aligned} \tag{12}$$

If $K_1 = n - 1$, then

$$|b_n|r^n = r^n = |a_{K_1}|r^{K_1+1}. \tag{13}$$

If $|z_1| = r$, then by (3) and (5), we have

$$|b_j|r^j < |a_{K_1}|r^{K_1+1}, \qquad \text{for } 0 \leq j < k_1 \text{ and for } K_1 + 1 < j \leq n.$$

Also, by (4), (7), (12), and (13), we have

$$|b_{k_1}|r^{k_1} \;=\; |b_{K_1+1}|r^{K_1+1} \;=\; |a_{K_1}|r^{K_1+1}.$$

Hence, taking (8) into account, $K = K_1 + 1$, while $k = k_1$, and so

$$K - k = K_1 - k_1 + 1,$$

as was to be shown.

If $|z_1| < r$, then by (3) and (5), we have

$$|b_j|r^j < |a_{K_1}|r^{K_1+1}, \qquad \text{for } 0 \le j < k_1 \text{ and for } K_1 + 1 < j \le n.$$

Also, by (4) and (6), we have

$$|b_{k_1}|r^{k_1} \;<\; |a_{K_1}|r^{K_1+1}.$$

Finally, by (12), (13), and (9), we have

$$|b_{K_1+1}|r^{K_1+1} \;=\; |a_{K_1}|r^{K_1+1} \;=\; |b_{k_1+1}|r^{k_1+1}.$$

Thus, again taking (8) into account, $K = K_1 + 1$, and $k = k_1 + 1$, so $K - k = K_1 - k_1$ as was to be shown.

For the last case in the induction, suppose $|z_1| > r$. By (3), (5), (11), and (13), we have

$$|b_j|r^j < \frac{|z_1|}{r}|a_{K_1}|r^{K_1+1} \qquad \text{for } j < k_1 \text{ and for } j > K_1.$$

By (8),

$$|b_j|r^j \le \frac{|z_1|}{r}|a_{K_1}|r^{K_1+1} \qquad \text{for } k_1 + 1 \le j \le K_1 - 1.$$

By (4), (7), and (10), we have

$$|b_{k_1}|r^{k_1} = |b_{K_1}|r^{K_1} = \frac{|z_1|}{r}|a_{K_1}|r^{K_1+1}.$$

Thus, $k = k_1$ and $K = K_1$, so $K - k = K_1 - k_1$ as required. □

2.4 Euclidean Division Algorithm

Following [Am], we analyze the Euclidean division algorithm for Laurent series.

Let $r \ge 0$. A polynomial P is called r**-dominant** if $K(P, r) = \deg P$ and it is called r**-extremal** if it is r-dominant and in addition $k(P, r) = 0$. Thus, a polynomial is r-dominant if and only if all of its zeros are located in $\mathbf{B}_{\le r}$, and a polynomial is r-extremal if and only if all of its zeros are located in the annulus $|z| = r$.

Lemma 2.4.1 (Continuity of Division). *Let $r > 0$ and let f be analytic on $\mathbf{B}_{\le r}$. Let P be a polynomial in $\mathbf{F}[z]$ with $P \not\equiv 0$, let $r > 0$, and assume that P is r-dominant. Then there exist a unique function q analytic on $\mathbf{B}_{\le r}$ and a unique polynomial R such that*

(i) $f = Pq + R$;

(ii) $\deg R < \deg P$;

(iii) $|R|_r \le |f|_r$; *and*

(iv) $|q|_r \le \dfrac{|f|_r}{|P|_r}$.

Remark. Lemma 2.4.1 is referred to as continuity of division because it implies that if f_1 and f_2 are analytic functions with $|f_1 - f_2|_r$ small, and if f_1 and f_2 are each divided by P to get quotients q_1 and q_2 and remainders R_1 and R_2, then $|q_1 - q_2|$ and $|R_1 - R_2|$ are both small.

Proof. We first prove the case when f is also a polynomial. In that case, the Euclidean algorithm gives unique polynomials q and R satisfying (i) and (ii), so it remains to check (iii) and (iv).

First consider the special case that $r = 1$ and all the coefficients of f and P have absolute value at most 1, *i.e.,* are elements of $\mathcal{O}$, and that at least one coefficient in each polynomial has absolute value 1. This means $|f|_1 = |P|_1 = 1$, and since P is 1-dominant, its leading coefficient must have absolute value 1. Then, the Euclidean division algorithm produces polynomials R and q with coefficients in $\mathcal{O}$ and since $|f|_1 = |P|_1 = 1$, (iii) and (iv) follow. Since multiplying f by a constant multiplies q and R by the same constant, the result continues to hold without the assumption that the coefficients of f are in $\mathcal{O}$. Multiplying P by a constant divides q by the same constant and does not change R, so the result continues to hold without the assumption that the coefficients of P are in $\mathcal{O}$ as well.

Still in the case that f is a polynomial, if r is in $|\mathbf{F}^\times|$, then by choosing a in $\mathbf{F}$ with $|a| = r$ and changing variables by replacing z with az, we reduce to the case above. If $|\ |$ is trivial, then the lemma is also trivial. When $|\ |$ is non-trivial, if r is not in $|\mathbf{F}^\times|$, then for r' in $|\mathbf{F}^\times|$ sufficiently close to r, we will have that P is r'-dominant, and the lemma follows since $|\ |_r$ is continuous in r.

If f is not a polynomial, we can find a sequence of polynomials f_n such that $|f - f_n|_r \to 0$, for instance by truncating the power series representation of f to higher and higher orders. Letting q_n and R_n be the quotients and remainders obtained by dividing the f_n by P, we have by the polynomial version of the lemma already proven that q_n and R_n are sequences in $\mathcal{A}[r]$ that are Cauchy sequences with respect to $|\ |_r$. Therefore they converge to q and R in $\mathcal{A}[r]$ by Proposition 2.2.4. As $\deg R_n < \deg P$, the R_n must converge to a polynomial, also of degree $< \deg P$. Properties (i)–(iv) are preserved under taking limits as $n \to \infty$. Property (iv) ensures that the quotient q is analytic on $\mathbf{B}_{\le r}$.

Now to check uniqueness in the general case, suppose we have

$$Pq_1 + R_1 = f = Pq_2 + R_2.$$

Then, we would have $P(q_1 - q_2) = R_2 - R_1$. Hence, if $q_1 \ne q_2$, then

$$K(R_2 - R_1, r) = K(P(q_1 - q_2), r) = K(P, r) + K(q_1 - q_2, r) = \deg P + K(q_1 - q_2, r) \ge \deg P,$$

which contradicts the fact that $R_2 - R_1$ is a polynomial of degree less than $\deg P$. □

Corollary 2.4.2. *Let $r_1 \le r \le r_2$ with $r > 0$, let f be in $\mathcal{A}[r_1, r_2]$, and let P be an r-extremal polynomial. Then, there exists a unique Laurent series q in $\mathcal{A}[r_1, r_2]$ and a unique polynomial R such that*

(i) $f = Pq + R$;

(ii) $\deg R < \deg P$;

(iii) $|R|_r \le |f|_r$; *and*

(iv) $|q|_r \le \dfrac{|f|_r}{|P|_r}$.

Proof. We begin by proving uniqueness. Suppose

$$Pq + R = P\tilde{q} + \widetilde{R}.$$

Then,

$$P(q - \tilde{q}) = \widetilde{R} - R.$$

Hence,

$$K(\widetilde{R}-R,r)=K(P,r)+K(q-\tilde{q},r)=\deg P+K(q-\tilde{q},r) \qquad \text{and}$$
$$k(\widetilde{R}-R,r)=k(P,r)+k(q-\tilde{q},r)=0+k(q-\tilde{q},r)$$

since P is r-extremal. Hence,

$$K(\widetilde{R}-R,r)-k(\widetilde{R}-R,r)\geq \deg P,$$

which contradicts the fact that $\widetilde{R}-R$ is a polynomial of degree $<\deg P$.

To prove existence, write

$$f(z)=\sum_{n=0}^{\infty}a_nz^n+\sum_{n=-\infty}^{-1}a_nz^n=f_+(z)+f_-(z).$$

First, apply the division algorithm to f_+ to get a power series q_+ and a polynomial R_+ such that $f_+=Pq_++R_+$. From the lemma, we know $\deg R_+<\deg P$,

$$|q_+|_r\leq\frac{|f_+|_r}{|P|_r} \qquad \text{and} \qquad |R_+|_r\leq|f_+|_r.$$

Since $r_2\geq r$, P is also r_2 dominant, and the lemma also implies that q_+ is in $\mathcal{A}[r_2]$. Next, observe that $z^{\deg P}P(z^{-1})$, the palindrome of P, is r^{-1}-extremal. Because $f_-(z^{-1})/z$ is a power series that converges for $|z|=r^{-1}$, we can apply the division algorithm to get a power series q_- and a polynomial R_- with $\deg R_-<\deg P$ such that

$$z^{\deg P}\frac{f_-(z^{-1})}{z}=z^{\deg P}P(z^{-1})q_-(z)+R_-(z).$$

Hence,

$$f_-(z)=P(z)\frac{q_-(z^{-1})}{z}+z^{\deg P-1}R_-(z^{-1}).$$

From the lemma, we also know that

$$|z^{\deg P-1}R_-(z^{-1})|_r=|z^{1-\deg P}R_-(z)|_{r^{-1}}\leq|f_-(z^{-1})|_{r^{-1}}=|f_-(z)|_r,$$

and

$$|z^{-1}q_-(z^{-1})|_r=|zq_-(z)|_{r^{-1}}\leq\frac{|f_-(z^{-1})|_{r^{-1}}}{|P(z^{-1})|_{r^{-1}}}=\frac{|f_-(z)|_r}{|P(z)|_r}.$$

Because $r_1\leq r$, $z^{\deg P}P(z^{-1})$ is r_1^{-1} dominant, and the lemma implies $z^{-1}q_-(z^{-1})$ is in $\mathcal{A}[r_1,\infty)$. Thus, the corollary follows by setting

$$q(z)=q_+(z)+z^{-1}q_-(z^{-1}) \qquad \text{and} \qquad R(z)=R_+(z)+z^{\deg P-1}R_-(z^{-1}). \quad \square$$

Weierstrass Preparation

The following theorem is a non-Archimedean one variable version of the Weierstrass Preparation Theorem and is sometimes referred to as a version of Hensel's Lemma.

Theorem 2.4.3 (Weierstrass Preparation). *Let f be an analytic function on $A[r_1,r_2]$. Let r be such that $r_1\leq r\leq r_2$. Let $d=K(f,r)-k(f,r)$. Then, there exists a unique pair (P,u) such that $f=Pu$, such that P is a polynomial of degree d with $P(0)=1$, $k(P,r)=0$, and $K(P,r)=d$, and such that u is analytic on $A[r_1,r_2]$ with $k(u,r)=K(u,r)$.*

Proof. Multiplying f by a constant and a suitable power of z, we may, without loss of generality, assume that $|f|_r = 1$ and $k(f,r) = 0$. We show existence by an inductive construction. Write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \qquad \text{and let} \qquad P_1(z) = \sum_{n=0}^{d} a_n z^n,$$

where $d = K(f,r)$. Note that P_1 is r-extremal, and let q_1 and R_1 be the quotient and remainder provided by Corollary 2.4.2. Observe that

$$f - P_1 = P_1(q_1 - 1) + R_1,$$

and by the uniqueness of the division algorithm and Corollary 2.4.2, we have

$$|R_1|_r \le |f - P_1|_r < 1,$$

keeping in mind that $|P_1|_r = 1$.

Now assume that for $i = 1, \ldots, n$ we have found (P_i, q_i, R_i) where the P_i are degree d r-extremal polynomials with $|P_i|_r = 1$, where the R_i are polynomials of degree $< d$, where the q_i are analytic on $A[r_1, r_2]$ with $|q_i|_r = 1$, where $f = P_i q_i + R_i$, and where the following inequalities hold:

(a) $|R_i|_r \le |f - P_1|^i$ for $i = 1, \ldots, n$;

(b) $|P_i - P_{i-1}|_r \le |f - P_1|^{i-1}$ for $i = 2, \ldots, n$; and

(c) $|q_i - q_{i-1}|_r \le |f - P_1|^i$ for $i = 2, \ldots, n$.

Set $P_{n+1} = P_n + R_n$. Now, P_{n+1} and P_n have the same top degree term and $|R_n|_r < 1 = |P_n|_r$, so $|P_{n+1}|_r = 1$ and P_{n+1} is r-dominant. Also,

$$|R_n(0)| \le |R_n|_r < 1 = |P_n|_r = |P_n(0)|,$$

and so by Exercise 1.1.1, $k(P_{n+1}, r) = 0$, and P_{n+1} is r-extremal. Let q_{n+1} and R_{n+1} be the quotient and remainder obtained by dividing f by P_{n+1}. Now,

$$|P_{n+1} - P_n|_r = |R_n|_r \le |f - P_1|_r^n,$$

and so (b) is satisfied by P_{n+1}.

Re-arranging the equation

$$P_n q_n + R_n = f = P_{n+1} q_{n+1} + R_{n+1} = (P_n + R_n) q_{n+1} + R_{n+1},$$

we get

$$-R_n q_{n+1} = P_n(q_{n+1} - q_n) + R_{n+1} - R_n \tag{14}$$

$$\text{and} \qquad R_n(1 - q_{n+1}) = P_n(q_{n+1} - q_n) + R_{n+1}. \tag{15}$$

Applying Corollary 2.4.2 to (14), we have

$$|q_{n+1} - q_n|_r \le |R_n|_r \cdot |q_{n+1}|_r \le |f - P_1|_r^n \cdot 1 < 1,$$

and hence $|q_{n+1}|_r = |q_n|_r = 1$ by Exercise 1.1.1. Now,

$$|1 - q_{n+1}|_r = |1 - q_1 + q_1 - q_2 + \cdots + q_n - q_{n+1}|_r \le \max\{|1 - q_1|_r, |q_1 - q_2|_r, \ldots, |q_n - q_{n+1}|_r\} \le |f - P_1|_r.$$

Combining this with applying Corollary 2.4.2 to (15), we get

$$|q_{n+1} - q_n|_r \le |R_n|_r \cdot |1 - q_n|_r \le |f - P_1|_r^n \cdot |f - P_1|_r = |f - P_1|_r^{n+1},$$

which shows (c), and (a) follows similarly.

By (a), $\lim R_n = 0$. Let $P = \lim P_n$ and $u = \lim q_n$. Then, $f = Pu$ and P is an r-extremal degree d polynomial since each P_n is. As u is the quotient under long division, Corollary 2.4.2 implies that u is in $\mathcal{A}[r_1, r_2]$. Because

$$d = K(f,r) - k(f,r) = K(P,r) - k(P,r) + K(u,r) - k(u,r) = d + K(u,r) - k(u,r),$$

we see that $K(u,r) = k(u,r)$, as was to be shown.

Only the uniqueness remains. Suppose $Pu = \widetilde{P}\tilde{u}$. By Proposition 2.3.3, u is invertible in $\mathcal{A}[r,r]$. Hence, $P = u^{-1}\widetilde{P}\tilde{u}$. However, $u^{-1}\tilde{u}$ is the quotient of P by $\widetilde{P}$, and is hence a polynomial, by the uniqueness of the quotient. Since P and $\widetilde{P}$ have the same degree, this implies $u^{-1}\tilde{u}$ is constant. Since $P(0) = \widetilde{P}(0) = 1$, that constant is 1 and we conclude $u = \tilde{u}$ and $P = \widetilde{P}$. □

Theorem 2.4.3 allows us to connect $|f|_r$, $K(f,r)$, $k(f,r)$, and the locations of the zeros of f.

Theorem 2.4.4. *Let f be analytic on $A[r_1, r_2]$. If $r_1 \le \rho \le R \le r_2$, then f has*

$$K(f,R) - k(f,\rho)$$

zeros in $A[\rho, R]$ counting multiplicity.

Proof. Clearly, $K(f,r)$ and $k(f,r)$ are non-decreasing in r, so it suffices to prove the theorem for $\rho = R = r$. Also, the case of $r = 0$ is clear, so we assume $r > 0$.

Write $f = Pu$ as in Theorem 2.4.3. From Proposition 2.3.4, P has

$$K(P,r) - k(P,r) = K(f,r) - k(f,r)$$

zeros, all with absolute value r. □

Theorem 2.4.4 tells us that the valuation polygon for a Laurent series f works just like the polynomial case. Namely, the zeros of f occur precisely at the corners of the valuation polygon, and the sharpness of the corner determines the number of zeros, counting multiplicity.

The connection between the locations of the zeros of a non-Archimedean analytic function and its Laurent series coefficients given by Theorem 2.4.4 is strikingly different from the classical complex case and is responsible for most of the differences between classical function theory and non-Archimedean function theory.

Corollary 2.4.5 (Identity Principle). *If $f \not\equiv 0$ is analytic on $A[r_1, r_2]$, with $r_2 < \infty$, then f has at most finitely many zeros in $A[r_1, r_2]$.*

Proof. The numbers $k(f,r)$ and $K(f,r)$ are non-decreasing functions of r, so the number of zeros is bounded by $K(f, r_2) - k(f, r_1)$. □

Theorem 2.4.4 allows us to immediately conclude non-Archimedean analogs of Picard's theorems for maps to the projective line.

Corollary 2.4.6 (Little Picard). *If f is analytic and zero free on $A[0,\infty)$, then f is constant.*

Proof. Since f is zero free, $f(0) \neq 0$, and so

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n, \quad \text{with } a_0 \neq 0.$$

Since we don't have any zeros, we have by Theorem 2.4.4, that

$$\sup_{n \ge 1} |a_n| r^n < |a_0| \quad \text{for all } r.$$

This is clearly impossible unless $a_n = 0$ for all $n \ge 1$. □

Corollary 2.4.7 (Big Picard). *If f is analytic and zero-free on $A[r_1, \infty)$, then*

$$\limsup_{r \to \infty} \frac{\log |f|_r}{\log r} < \infty.$$

In particular, f is meromorphic at infinity.

Remark. Corollary 2.4.7 seems to have first appeared in the literature in [vdP], although van der Put himself says that this surely must have been known much earlier.

Proof. Write

$$f(z) = \sum_{n \in \mathbf{Z}} a_n z^n.$$

Because f is zero free,

$$|a_n| r^n < |f|_{r_1} \quad \text{for } |n| \text{ large.}$$

Thus, f clearly has only finitely many non-zero a_n with $n > 0$. The proof is completed by Proposition 2.2.8. □

2.5 Poisson-Jensen Formula

We conclude this lecture with a non-Archimedean Poisson-Jensen formula. Let f be an analytic function on $A[r_1, r_2]$ which is not identically zero. We define the **counting function** $N(f, 0, r)$ by defining

$$N(f, 0, r) = \sum_{\substack{0 \neq z \in A[r_1, r] \\ \text{s.t. } f(z) = 0}} \log \frac{r}{|z|}.$$

Here, we count the zeros of $f(z)$ with multiplicity. If $r_1 = 0$, it is convenient to add the term $K(f, 0) \log r$ to the definition of $N(f, 0, r)$. By the identity principle, the sum defining N is finite if $r \in [r_1, r_2]$. *Note that N implicitly depends on the lower radius in the annulus r_1.*

Theorem 2.5.1 (Poisson-Jensen). *Let f be a non-constant analytic function on $A[r_1, r_2)$, with $r_2 \leq \infty$. Let*

$$f(z) = \sum_{n \in \mathbf{Z}} a_n z^n$$

be the Laurent expansion for f. Then, for all $r \in [r_1, r_2)$, we have

$$N(f, 0, r) + k(f, r_1) \log r + \log |a_{k(f, r_1)}| = \log |f|_r \qquad \text{if } r_1 > 0$$

or

$$N(f, 0, r) + \log |a_{K(f,0)}| = \log |f|_r \qquad \text{if } r_1 = 0.$$

Remark. If $r_1 = 0$ or $r_2 < \infty$, then the theorem says the difference between $N(f, 0, r)$ and $\log |f|_r$ remains bounded as $r \to r_2$. If $r_1 > 0$ and $r_2 = \infty$, then the difference is bounded by $O(\log r)$ as $r \to \infty$.

Proof. This is basically unwinding definitions and understanding the valuation polygon. Recall that by Proposition 2.3.1, there are only finitely many critical points in $[r_1, r]$ if $r < r_2$.

In the case $r_1 = 0$, let r' be the smallest positive critical point. By Theorem 2.4.3, f has no zeros with absolute value between 0 and r'. Thus, for $0 < r \leq r'$,

$$\begin{aligned} N(f, 0, r) \ = \ K(f, 0) \log r \ &= \ K(f, r) \log r \\ &= \ \log |f|_r - \log |a_{K(f,r)}| \ = \ \log |f|_r - \log |a_{K(f,0)}|, \end{aligned}$$

where we have used $K(f, r) = K(f, 0)$.

In the case $r_1 > 0$, let r' be the smallest critical point $> r_1$. Again, by Theorem 2.4.3, f has no zeros with absolute value between r_1 and r'. Therefore, for $r_1 \le r \le r'$, again using $K(f,r) = K(f,r_1)$, and the fact that

$$\log|a_{k(f,r_1)}| + k(f,r_1)\log r_1 = \log|f|_{r_1} = \log|a_{K(f,r_1)}| + K(f,r_1)\log r_1, \tag{16}$$

we get

$$\begin{aligned}
N(f,0,r) &= \sum_{\substack{|z|=r_1 \\ \text{s.t. } f(z)=0}} \log\frac{r}{|z|} \\
&= [K(f,r_1) - k(f,r_1)]\log\frac{r}{r_1} \\
&= K(f,r_1)\log r - k(f,r_1)\log r + k(f,r_1)\log r_1 - K(f,r_1)\log r_1 \\
\text{[From (16)]} \quad &= K(f,r)\log r - k(f,r_1)\log r + \log|a_{K(f,r_1)}| - \log|a_{k(f,r_1)}| \\
&= \log|f|_r - \log|a_{K(f,r)}| - k(f,r_1)\log r \\
&\qquad + \log|a_{K(f,r_1)}| - \log|a_{k(f,r_1)}| \\
&= \log|f|_r - k(f,r_1)\log r - \log|a_{k(f,r_1)}|.
\end{aligned}$$

Thus, in both cases, we see that the desired formula is correct for r between r_1 and the first critical point bigger than r_1. We then simply need to check that we can pass through each critical point. Thus, assume r' is a critical point. Assume the formula of the theorem is true for $r \le r'$. Let r'' be the smallest critical point larger than r'. As there are at most finitely many critical points between r_1 and any $r < r_2$, we simply need to show the formula remains valid for $r' < r \le r''$, and the theorem follows by induction. Indeed, as above,

$$\begin{aligned}
N(f,0,r) - N(f,0,r') &= \sum_{\substack{|z|=r' \\ \text{s.t. } f(z)=0}} \log\frac{r}{r'} \\
&= [K(f,r') - k(f,r')]\log\frac{r}{r'} \\
&= \log|f|_r - k(f,r')\log r - \log|a_{k(f,r')}| \\
&= \log|f|_r - \log|f|_{r'}.
\end{aligned}$$

□

To keep the focus on the essential ideas, I will confine myself to a discussion of one variable in these lectures. The several variable theory is also well-developed. In particular, one can consider multivariable power series or Laurent series and define $|\ |_r$ similarly to what was done in today's lecture. One of the contributions in [CY 1] is a several variable Poisson-Jensen formula which shows that also in several variables $\log|f|_r$ measures the quantity of zeros of f in an appropriate sense. Only power series are discussed in [CY 1], but one can also work with multivariable Laurent series, as discussed in [ChRu]. Some additional discussion of several variables, particularly in postive characteristic, is contained in [ChTo].

3 Non-Archimedean Value Distribution Theory

In this lecture we introduce the non-Archimedean analog of Nevanlinna's theory of value distribution.

3.1 Nevanlinna's Theory of Value Distribution

In a deep and beautiful theory, Nevanlinna developed quantitative analogs of the Fundamental Theorem of Algebra for meromorphic functions. His theory continues to be an indispensable tool in, for example, the study of complex dynamics and the study of meromorphic solutions to differential equations; see *e.g.* [Brg 1] and [Lai].

Given a meromorphic function f, given a value a in $\mathbf{P}^1(\mathbf{C})$, and given a radius r, Nevanlinna introduced the following functions. The **counting function** $N(f,a,r)$ counts, as a logarithmic average, the number of times f takes on the value a in the disc of radius r, and is precisely defined by

$$N(f,a,r) = \sum_{\substack{0<|z|<r \\ \text{s.t. } f(z)=a}} \log\frac{r}{|z|} \;+\; m\log r,$$

where values z in the sum over $f(z)=a$ are repeated, according to their multiplicity, and where m is the order of vanishing of $f(z)-a$ at the origin, or of $1/f(z)$ if $a=\infty$. One can also define the **truncated counting function** $N^{(1)}(f,a,r)$, which counts the zeros without regard to multiplicity.

The **proximity function** $m(f,a,r)$ measures how close the function f stays to a on the circle of radius r and is defined by

$$m(f,a,r) = \int_0^{2\pi} \log^+\left|\frac{1}{f(re^{i\theta})-a}\right|\frac{d\theta}{2\pi},$$

where $\log^+ x$ denotes $\max\{0,\log x\}$. If $a=\infty$, then

$$m(f,\infty,r) = \int_0^{2\pi} \log^+|f(re^{i\theta})|\,\frac{d\theta}{2\pi}.$$

Nevanlinna's **characteristic function** $T(f,a,r) = m(f,a,r)+N(f,a,r)$ is the sum of the counting and proximity functions. Nevanlinna then proved two main theorems.

Theorem 3.1.1 (Nevanlinna's First Main Theorem). *If f is a non-constant meromorphic function on $\mathbf{C}$ and a is a point in $\mathbf{C}$, then $T(f,a,r)-T(f,\infty,r)$ remains bounded as $r\to\infty$.*

Remark. Nevanlinna's First Main Theorem says that $m(f,a,r)+N(f,a,r)$ is, up to a bounded term, independent of a as $r\to\infty$. This has two consequences. First, since $m(f,a,r)\geq 0$, it gives an upper-bound on the frequency with which f can take on the value a. In this sense, the First Main Theorem is an analog of the fact that a polynomial of degree d takes on the value a at most d times. The second consequence is that if f takes on a value a relatively rarely, then it must compensate for it by remaining close to the value a on a large proportion of each circle centered at the origin. Considering the function e^z is instructive. For values a other than 0 and ∞, e^z, being periodic, takes on each value with the same frequency. However, the values 0 and ∞ are omitted entirely. On the other hand, e^z is close to 0 and close to ∞ on about half of each circle centered at the origin, whereas it is only rarely close to any non-zero value a.

Theorem 3.1.2 (Nevanlinna's Second Main Theorem). *Let $a_1,\ldots,a_q$ be q distinct points in $\mathbf{P}^1(\mathbf{C})$, and let f be a non-constant meromorphic function on $\mathbf{C}$. Then,*

$$(q-2)T(f,\infty,r) - \sum_{j=1}^{q} N^{(1)}(f,a_j,r) \leq O(\log T(f,r))$$

as $r\to\infty$ outside an exceptional set of radii of finite Lebesgue measure.

Remark. When $q>2$, the inequality in the Second Main Theorem gives a lower bound on the sum of the counting functions, so f cannot take on too many values with lower than expected frequency.

Detailed introductions to Nevanlinna's theory can be found in [Nev], [Hay], and [CY 2].

This lecture will introduce the analog of Nevanlinna's theory in non-Archimedean function theory. Perhaps the earliest work with the spirit of non-Archimedean Nevanlinna theory was the work of Adams and Straus [AdSt]. Hà Huy Khoái, one of the organizers of this school, was the first to set out to systematically develop a complete analog of Nevanlinna's theory for non-Archimedean meromorphic functions [Kh 1].

3.2 Prescribing Zeros to Analytic Functions

The following theorem is a well-known consequence of the Mittag-Leffler theorem and is often covered in a first course in complex analysis.

Theorem 3.2.1 (Mittag-Leffler). *Let D be a domain in $\mathbf{C}$, let z_n be a discrete sequence of distinct points in D, and let $m_n \geq 1$ be a sequence of positive integers. Then, there exists an analytic function on D such that for each n, f has a zero of multiplicity m_n at z_n and such that f has no other zeros.*

We now explore some non-Archimedean analogs of this theorem. First, if f is analytic in $A[r_1, r_2]$, then by Corollary 2.4.5, f has only finitely many zeros in $A[r_1, r_2]$. Conversely, by taking for instance a polynomial, given a finite number of points in $A[r_1, r_2]$ and associated multiplicities, we can construct a function f analytic on $A[r_1, r_2]$ with the prescribed zeros and multiplicities.

Theorem 3.2.2. *Let z_n be a sequence of distinct non-zero numbers in $\mathbf{F}$ such that $|z_n| \to \infty$. Let m_n be a sequence of positive integers, and let m_0 be a non-negative integer. Then,*

$$f(z) = z^{m_0} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right)^{m_n}$$

converges to an analytic function f on $\mathbf{F}$ with a zero at 0 of multiplicity m_0, with zeros at z_n with multiplicities m_n, and with no other zeros.

Lemma 3.2.3. *Let f_n be analytic functions on $A[r, r]$. Then, $\prod f_n$ converges if and only if* $\lim f_n = 1$.

Proof. Consider the partial products,

$$P_N = \prod_{n=1}^{N} f_n.$$

We need to check that $|P_N - P_M|_r$ tends to zero as $\min\{N, M\} \to \infty$. Without loss of generality, assume that $N \geq M$. Then,

$$|P_M - P_N|_r = \left|\prod_{n=1}^{M} f_n\right|_r \left|1 - \prod_{n=M+1}^{N} f_n\right|_r .$$

Because $|1 - f_n|_r \to 0$, there exists an N_0 such that for all $n \geq N_0, |f_n|_r = 1$. Therefore for $M \geq N_0$,

$$\left|\prod_{n=1}^{M} f_n\right|_r = \left|\prod_{n=1}^{N_0} f_n\right|_r .$$

On the other hand, if $M \geq N_0$, then

$$\left|1 - \prod_{n=M+1}^{N} f_n\right|_r = \left|1 - \prod_{n=M+1}^{N} (1 - (1 - f_n))\right|_r \leq \sup_{n>M} |1 - f_n|_r.$$

The right hand side tends to zero as $M \to \infty$ by assumption, and the proposition follows. □

Proof of Theorem 3.2.2. The product in the statement of the theorem converges to an entire function by Lemma 3.2.3. The product clearly has the prescribed zeros with the prescribed multiplicities. Fix $r > 0$. By Proposition 2.3.3, the product over all z_n with $|z_n| > r$ converges to a unit in $\mathcal{A}[r]$. Hence, f only has the zeros prescribed by the product. □

The situation for finite non-bordered discs is more delicate. A complete non-Archimedean field is called **maximally complete** or **spherically complete** if every collection of embedded discs $D_{i+1} \subset D_i$ has non-empty intersection.

Exercise 3.2.4. *The field* $\mathbf{C}_p$ *is not maximally complete.*

Theorem 3.2.5 (Lazard). *The following are equivalent:*

(a) The field $\mathbf{F}$ *is maximally complete.*

(b) Given $R > 0$*, given a sequence of distinct points* z_n *in* $\mathbf{B}_{<R}$ *such that* $|z_n| \to R$*, and given positive integers* m_n*, there exists an analytic function* f *in* $\mathcal{A}(R)$ *with a zero at each* z_n *with multiplicity* m_n *and no other zeros.*

The proof of Theorem 3.2.5 will not be discussed here. See [Laz].

3.3 Nevanlinna Functions and the First Main Theorem

As in the classical complex case, the non-Archimedean First Main Theorem is simply the Poisson-Jensen formula dressed up in new notation.

The non-Archimedean counting functions are defined in exactly the same way as in the complex case. If f is a meromorphic function, the proximity function is defined by

$$m(f, a, r) = \log^+ \left| \frac{1}{f-a} \right|_r \qquad \text{and} \qquad m(f, \infty, r) = \log^+ |f|_r.$$

The characteristic function can then be defined, just as over the complex numbers, by

$$T(f, a, r) = m(f, a, r) + N(f, a, r).$$

Theorem 3.3.1 (First Main Theorem). *If* f *is a non-constant meromorphic function on* $\mathbf{F}$*, then* $T(f, a, r) - T(f, \infty, r)$ *remains bounded as* $r \to \infty$*.*

Proof. We first treat the case that $a = 0$. By Theorem 3.2.2, we can write $f = g/h$, where g and h are entire without common zeros. The Poisson-Jensen Formula (Theorem 2.5.1) then tells us that

$$N(f, \infty, r) = N(h, 0, r) = \log |h|_r + O(1) \qquad \text{and} \qquad N(f, 0, r) = N(g, 0, r) = \log |g|_r + O(1),$$

and so

$$N(f, \infty, r) - N(f, 0, r) = \log |h|_r - \log |g|_r + O(1).$$

Now,

$$m(f, \infty, r) = \max\{0, \log |g|_r - \log |h|_r\} \qquad \text{and} \qquad m(f, 0, r) = \max\{0, \log |h|_r - \log |g|_r\}.$$

Thus,

$$m(f, \infty, r) - m(f, 0, r) = \log |g|_r - \log |h|_r,$$

which proves the theorem when $a = 0$.

If $a \neq 0$, then $N(f, a, r) = N(f - a, 0, r)$ and $N(f, \infty, r) = N(f - a, \infty, r)$. Also,

$$m(f, a, r) = m(f - a, 0, r) \qquad \text{and} \qquad m(f, \infty, r) = m(f - a, \infty, r) + O(1). \quad \square$$

3.4 Hasse Derivatives

We now examine Hasse derivatives more closely. First note that if n is a non-negative integer and k is an integer not in the interval $[0, n-1]$, then the binomial coefficient

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

is well-defined and non-zero. Thus, we can extend the notion of Hasse derivatives to Laurent series as follows. If

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \qquad \text{then define} \qquad D^n f(z) = \sum_{k \in \mathbf{Z}\setminus[0,n-1]} \binom{k}{n} a_k z^{k-n},$$

and again if $\mathbf{F}$ has characteristic zero, then $D^n f = f^{(n)}/n!$. By defining $\binom{k}{n}$ to be zero if $0 \le k \le n-1$, we can write this simply as

$$D^n f(z) = \sum_{k=-\infty}^{\infty} \binom{k}{n} a_k z^{k-n}.$$

Proposition 3.4.1. *The Hasse derivatives of analytic funcions on annuli satisfy the following basic properties:*

(i) $D^k[f+g] = D^k f + D^k g$;

(ii) $D^k[fg] = \displaystyle\sum_{i+j=k} D^i f D^j g$;

(iii) $D^i D^j f = \dbinom{i+j}{j} D^{i+j} f$.

(iv) *If* $\mathbf{F}$ *has positive characteristic* p *and* $s \ge 0$ *is an integer, then* $D^{p^s} f^{p^s} = (D^1 f)^{p^s}$.

Proof. Property (i) is obvious. To check property (ii), write out both sides and compare like powers of z. What is needed for equality is that for ℓ and m integers and i and j non-negative integers, one has

$$\sum_{i+j=k} \binom{\ell}{i}\binom{m}{j} = \binom{\ell+m}{k},$$

which is nothing other than Vandermonde's Identity. To check property (iii), one needs the elementary identity

$$\binom{i+j}{j}\binom{k}{i+j} = \binom{k}{j}\binom{k-j}{i}.$$

What one needs for (iv) is that fact that for any integer j,

$$\binom{jp^s}{p^s} \equiv j \bmod p,$$

which follows immediately from Lucas's Theorem. $\square$

Property (ii) in Proposition 3.4.1 allows us to inductively extend $D^n f$ to meromorphic functions f. For example,

$$D^1(f) = D^1\left[\frac{f}{g} g\right] = gD^1\left(\frac{f}{g}\right) + \frac{f}{g} D^1 g,$$

and hence

$$D^1\left(\frac{f}{g}\right) = \frac{gD^1 f - fD^1 g}{g^2}$$

as expected.

3.5 Logarithmic Derivative Lemma

Nevanlinna's first proof of his Second Main Theorem is based on a deep property of logarithmic derivatives, namely that if f is a meromorphic function on $\mathbf{C}$, then $m(f'/f,\infty,r)$ is small relative to $T(f,\infty,r)$ outside a small exceptional set of radii r; we will not give the precise formulation here for the complex numbers.

The non-Archimedean analog of the Logarithmic Derivative Lemma is a trivial, but useful, observation.

Lemma 3.5.1 (Logarithmic Derivative Lemma). *Let f be a non-Archimedean meromorphic function on the annulus $\{z : r_1 \le |z| \le r_2\}$. Then, for $r_1 \le r \le r_2$,*

$$\left|\frac{D^n f}{f}\right|_r \le \frac{1}{r^n}.$$

Here $D^n f$ denotes the n-th Hasse derivative of f.

Proof. We first prove the theorem in the case that f is analytic. Write f as a Laurent series $f(z)=\sum c_k z^k$. Then,

$$D^n f(z) = \sum_{k\in\mathbf{Z}} c_k \binom{k}{n} z^{k-n}.$$

Because the binomial coefficients are integers, they have absolute value ≤ 1, and so

$$|D^n f|_r = \sup_{k\in\mathbf{Z}} |c_k| \left|\binom{k}{n}\right| r^{k-n} \le \frac{\sup_{k\in\mathbf{Z}} |c_k| r^k}{r^n} = \frac{|f|_r}{r^n},$$

and hence the lemma is proven for analytic f.

We prove the lemma for meromorphic f by induction on n, the case that $n=0$ being trivial. Write $f=g/h$, where g and h are analytic. Using property (ii) of Proposition 3.4.1 for the Hasse derivative extended to meromorphic functions, we find that

$$\frac{D^{n+1}f}{f} = \frac{D^{n+1}(g/h)}{g/h} = \frac{D^{n+1}g}{g} - \frac{D^n f}{f}\cdot\frac{D^1 h}{h} - \cdots - \frac{D^1 f}{f}\cdot\frac{D^n h}{h} - \frac{D^{n+1}h}{h},$$

and so the lemma follows by induction, by the analytic case proven above, and by the fact that $|\ |_r$ is a non-Archimedean absolute value. □

An Application

We recall here a cute argument, sometimes atributed to Gauss, that if f, g are two complex polynomials in $\mathbf{C}[z]$, not both constant such that $f^{-1}(0)=g^{-1}(0)$ and such that $f^{-1}(1)=g^{-1}(1)$, then $f=g$. To prove this, assume $\deg f \ge \deg g$, and consider the rational function

$$h = \frac{f'(f-g)}{f(f-1)}.$$

Now, the degree of the denominator is twice the degree of f and the degree of the numerator is strictly less than twice the degree of f. On the other hand, if z_0 were a pole of h, then $f(z_0)=0$ or 1, and so by assumption $f-g$ vanishes at z_0. The presence of f' in the numerator ensures that h has no multiple poles, and thus h has no poles at all. Since the degree of the denominator is larger than that of the numerator, this means h is identically zero. Since f is non-constant, f' is not identically zero, and so we must have $f=g$. Lemma 3.5.1 allows us to make this same argument for non-Archimedean entire functions.

Proposition 3.5.2 (Adams & Straus). *Let f and g be non-Archimedean entire functions on $\mathbf{F}$, not both constant and if $\mathbf{F}$ has positive characteristic p such that neither f nor g is a pure p-th power. Let a and b be two distinct points in $\mathbf{F}$. If $f^{-1}(a) = g^{-1}(a)$ and $f^{-1}(b) = g^{-1}(b)$, then $f = g$.*

Proof. Without loss of generality, assume there is a sequence of $r_n \to \infty$ such that $|f|_{r_n} \geq |g|_{r_n}$. The hypotheses imply that $|f|_{r_n} \to \infty$. As in the polynomial case, let

$$h = \frac{f'(f-g)}{(f-a)(f-b)}.$$

Then, h has no poles, and is therefore entire. On the other hand,

$$|h|_{r_n} = \left|\frac{(f-a)'}{f-a}\right|_{r_n} \cdot \frac{|f-g|_{r_n}}{|f-b|_{r_n}} \leq \frac{1}{r_n},$$

for r_n large enough that $|f|_{r_n} > |b|$, and hence $h \equiv 0$. □

Remark. Lemma 3.5.1 implies $\log |f'/f|_r \leq -\log r$. When $r \geq 1$, the right-hand-side is bounded as $r \to \infty$. In the early works on non-Archimedean Nevanlinna theory, people, including me, sometimes replaced the $-\log r$ term with $O(1)$. Khoái and Tu's work [KhTu] made clear that keeping the $-\log r$ term is essential for applications such as the above.

3.6 Second Main Theorem

Second Main Theorem without Ramification

We stated Nevanlinna's Second Main Theorem over the complex numbers as an inequality involving

$$\sum_{j=1}^{q} N^{(1)}(f, a_j, r),$$

a sum of truncated counting functions. Of course this inequality implies the weaker inequality where the above sum is replaced by

$$\sum_{j=1}^{q} N(f, a_j, r),$$

the sum of non-truncated counting functions. This weaker inequality over the complex numbers is still much deeper than Nevanlinna's First Main Theorem. However, in the non-Archimedean case, Ru [Ru] made the significant observation that the non-Archimedean Second Main Theorem without ramification (or truncation) is a simple consequence of the First Main Theorem.

Theorem 3.6.1 (Second Main Theorem without Ramification). *Let $a_1, \ldots, a_q$ be q distinct points in $\mathbf{P}^1(\mathbf{F})$. Then,*

$$(q-1)T(f, \infty, r) - \sum_{j=1}^{q} N(f, a_j, r) \leq O(1)$$

as $r \to \infty$.

Before proceeding to the proof, several comments are in order. First observe that in the non-Archimedean case we have $(q-1)$ on the left hand side rather than $(q-2)$ as in the complex case. Thus, the stronger non-Archimedean Picard theorem follows: namely, a non-Archimedean meromorphic function on $\mathbf{F}$ can omit at most one value in $\mathbf{P}^1(\mathbf{F})$. Second, there is no need for an exceptional set of radii r in the non-Archimedean case.

Proof. First assume all the a_i are finite and let $d = \min_{i \neq j} |a_i - a_j| > 0$. Then, given $i \neq j$ and $r > 0$,

$$d \leq |a_j - a_i| = |a_j - a_i|_r = |(f - a_i) - (f - a_j)|_r \leq \max\{|f - a_i|_r, |f - a_j|_r\}.$$

This implies that given $r > 0$, there is at most one index j_0 such that $|f - a_{j_0}|_r < d$. Thus,

$$\sum_{j \neq j_0} [T(f, a_j, r) - N(f, a_j, r)] = \sum_{j \neq j_0} m(f, a_j, r) = \sum_{j \neq j_0} \log^+ |f - a_j|_r^{-1} \leq (q-1) \log^+(1/d).$$

By the First Main Theorem, up to a bounded term, we can replace

$$\sum_{j \neq j_0} T(f, a_j, r)$$

with $(q-1)T(f, \infty, r)$. Since $N(f, a_{j_0}, r) \geq 0$ for $r \geq 1$, we can subtract $N(f, a_{j_0}, r)$ from the left-hand side to get the theorem in the case that none of the a_j are infinite. If one of the $a_j = \infty$, let c be a point of $\mathbf{P}^1(\mathbf{F})$ that is not among the a_j. Consider the meromorphic function $g = (f - c)^{-1}$ and let $b_j = (a_j - c)^{-1}$. Then, the b_j are all finite and we can apply what we've already proven to g and the b_j. Of course $N(g, b_j, r) = N(f, a_j, r)$, and by the First Main Theorem, $T(f, \infty, r) = T(g, \infty, r) + O(1)$. □

The Defect Relation

The **defect** of a point a in $\mathbf{P}^1(\mathbf{F})$ with respect to a meromorphic function f is defined by

$$\delta_f(a) = \liminf_{r \to \infty} \frac{m(f, a, r)}{T(f, \infty, r)} = 1 - \limsup_{r \to \infty} \frac{N(f, a, r)}{T(f, \infty, r)}.$$

The Second Main Theorem immediately implies:

Corollary 3.6.2. $\displaystyle\sum_{a \in \mathbf{P}^1(\mathbf{F})} \delta_f(a) \leq 1.$

In fact, something much stronger is true.

Proposition 3.6.3. *Given a non-constant meromorphic function f on $\mathbf{F}$, there is at most one point a in $\mathbf{P}^1(\mathbf{F})$ such that $\delta_f(a) > 0$.*

Proof. By making a projective change of coordinates if necessary, without loss of generality assume, $\delta_f(0) > 0$ and $\delta_f(\infty) > 0$. Then, $\delta_f(\infty) > 0$ and $\delta_f(0) > 0$ imply that $m(f, 0, r)$ and $m(f, \infty, r)$ are both positive for all sufficiently large r. But this exactly means that both $|f|_r > 1$ and $|1/f|_r > 1$, which is clearly absurd. □

As $\delta_f(a) \leq 1$, Proposition 3.6.3 is stronger than Corollary 3.6.2. Given a in $\mathbf{P}^1(\mathbf{F})$ and $0 < \delta \leq 1$, there exists a meromorphic function f on $\mathbf{F}$ such that $\delta_f(a) = \delta$; see, *e.g.* [CY 1].

The Second Main Theorem with Ramification

Define,

$$N_{\mathrm{Ram}}(f, r) = N(f', 0, r) + 2N(f, \infty, r) - N(f', \infty, r).$$

Observe that $N_{\mathrm{Ram}}(f, r)$ exactly counts the ramification points of f with multiplicity, and that by Poisson-Jensen, we can also write

$$N_{\mathrm{Ram}}(f, r) = 2N(f, \infty, r) + \log |f'|_r + O(1). \tag{17}$$

Theorem 3.6.4 (Second Main Theorem with Ramification). *Let f be meromorphic on $\mathbf{F}$ and assume $f' \not\equiv 0$. Let $a_1, \dots, a_q$ be q distinct points in $\mathbf{P}^1(\mathbf{F})$. Then,*

$$(q-2)T(f,\infty,r) - \sum_{j=1}^{q} N^{(1)}(f,a_j,r) \le (q-2)T(f,\infty,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{\mathrm{Ram}}(f,r) \le -\log r + O(1).$$

Proof. The first inequality is clear, so we show the second. Let q' denote the number of finite a_j, and if $q' < q$, without loss of generality, assume $a_q = \infty$. As in the proof of Theorem 3.6.1, let $d = \min_{i \ne j} |a_i - a_j|$, where the minimum is taken over the finite a_i and a_j. For the moment, fix $r > 0$. Again, we see that there is an index j_0, which may depend on r, such that for all $j \ne j_0$ and $j \le q'$, we have $|f - a_j|_r \ge d$. Thus,

$$\begin{aligned}(q'-1)m(f,\infty,r) = (q'-1)\log^+|f|_r &\le \sum_{j \ne j_0} \log|f - a_j|_r + (q'-1)\log^+\frac{1}{d} + (q'-1)\max_{1 \le j \le q'} |a_j| \\ &= \sum_{j \ne j_0} \log|f - a_j|_r + O(1),\end{aligned}$$

where the $O(1)$ term is independent of r and j_0, and the sum over $j \ne j_0$ means the sum over all indices $\le q'$ and different from j_0. Now,

$$\sum_{j \ne j_0} \log|f - a_j|_r \le \sum_{j=1}^{q'} \log|f - a_j|_r - \log|f'|_r + \log\left|\frac{f'}{f - a_{j_0}}\right|_r \le \sum_{j=1}^{q'} \log|f - a_j|_r - \log|f'|_r - \log r$$

by the Logarithmic Derivative Lemma. Hence,

$$(q'-1)m(f,\infty,r) \le \sum_{j=1}^{q'} \log|f - a_j|_r - \log|f'|_r - \log r + O(1).$$

Since the right-hand-side does not depend on j_0, we no longer need to regard r as fixed. Now we apply Poisson-Jensen to get

$$(q'-1)m(f,\infty,r) \le \sum_{j=1}^{q'} N(f,a_j,r) - q'N(f,\infty,r) + 2N(f,\infty,r) - N_{\mathrm{Ram}}(f,r) - \log r + O(1).$$

Hence,

$$(q'-1)T(f,\infty,r) - \sum_{j=1}^{q'} N(f,a_j,r) - N(f,\infty,r) + N_{\mathrm{Ram}}(f,r) \le -\log r + O(1).$$

This is precisely the statement of the theorem when $a_q = \infty$. When $q = q'$, the theorem follows by replacing $N(f,\infty,r)$ on the left with the larger $T(f,\infty,r)$. □

The Second Main Theorem was first proven in characteristic zero by C. Corrales-Rodrigáñez [Co 1], but her work only became easily accessible in the literature some years later [Co 2]. The Second Main Theorem was also proven independently by Khoái and Quang [KhQu] and by Boutabaa [Bo 2]. Of course the same proof works in positive characteristic, provided f is not a p-th power. The formulation above sufficies for all applications I know of. One can state an inequality valid for all non-constant functions in positive characteristic that essentially amounts to the following:

Corollary 3.6.5. *Let $\mathbf{F}$ have positive characteristic, let f be meromorphic on $\mathbf{F}$ such that $f' \not\equiv 0$, let $a_1, \dots, a_q$ be q distinct points in $\mathbf{P}^1(\mathbf{F})$, and let s be a non-negative integer. Then,*

$$(q-2)T(f^{p^s},\infty,r) - \sum_{j=1}^{q} N(f^{p^s}, a^{p^s}, r) + p^s N_{\mathrm{Ram}}(f,r) \le -p^s \log r + O(1).$$

This observation, together with the fact that $N_{\mathrm{Ram}}(f,r)$ cancels any contribution to the counting functions $N(f^{p^s},a^{p^s},r)$ coming from points whose multiplicity is divisible by p^{s+1}, is essentially the content of [BoEs].

The ABC Inequality

Corollary 3.6.6 (ABC). *Let $f+g=h$ be relatively prime entire functions, not all of whose derivatives vanish identically. Then,*

$$\log\max\{|f|_r,|g|_r,|h|_r\} \le N^{(1)}(fgh,0,r)-\log r+O(1).$$

Proof. Let $F=f/h$. By the relatively prime assumption,

$$N^{(1)}(fgh,0,r)=N^{(1)}(F,0,r)+N^{(1)}(F,\infty,r)+N^{(1)}(F,1,r).$$

Applying the Second Main Theorem,

$$N^{(1)}(fgh,0,r)-\log r+O(1)\ge T(F,\infty,r)=\log^+\left|\frac{f}{h}\right|_r+N(h,0,r).$$

Now, $N(h,0,r)=\log|h|_r+O(1)$ by Poisson-Jensen, so

$$\log^+\left|\frac{f}{h}\right|_r+N(h,0,r)=\log\max\{|f|_r,|h|_r\}+O(1).$$

Since $g=h-f$, we know $|g|_r\le\max\{|f|_r,|h|_r\}$, and the corollary follows. □

Hu and Yang undertook a systematic study of generalized ABC-inequalities for non-Archimedean entire functions: [HY 2], [HY 3], and [HY 4]. See [ChTo] for positive characteristic.

The Defect Relation Again

The **ramification defect** $\theta_f(a)$ of a point a in $\mathbf{P}^1(\mathbf{F})$ with respect to a meromorphic function f is defined by

$$\theta_f(a)=1-\limsup_{r\to\infty}\frac{N^{(1)}(f,a,r)}{T(f,\infty,r)}.$$

The Second Main Theorem immediately implies

Corollary 3.6.7. *If f is a meromorphic function on $\mathbf{F}$ such that $f'\not\equiv 0$, then*

$$\sum_{a\in\mathbf{P}^1(\mathbf{F})}\theta_f(a)\le 2,$$

with strict inequality if f is a rational function.

Unlike the case with ordinary defects, it is possible for $\theta_f(a)$ to be positive for more than one value of a. Not much is known about ramification defects for non-Archimedean meromorphic functions, and it would be interesting to say anything non-trivial about them.

Some Applications

A value a is called **totally ramified** for a meromorphic function of f if every point in $f^{-1}(a)$ is a ramification point. For example, 1 and -1 are totally ramified values for the sine and cosine functions.

Corollary 3.6.8. *If f is a non-Archimedean meromorphic function such that $f'\not\equiv 0$, then f has at most three totally ramified values.*

Proof. Suppose $a_1, \ldots, a_4$ are totally ramified values. Then, the Second Main Theorem says:

$$2T(f,\infty,r) - \sum_{j=1}^{4} N^{(1)}(f,a_j,r) \leq -\log r + O(1).$$

But because the a_j are totally ramified,

$$N^{(1)}(f,a_j,r) \leq \frac{1}{2}N(f,a_j,r) \leq \frac{1}{2}T(f,\infty,r) + O(1),$$

where the second inequality follows from the First Main Theorem. We thus conclude $\log r \leq O(1)$ and thereby reach a contradiction. $\square$

It is easy to check that 0, 1, and ∞ are totally ramified values for the rational function

$$f(z) = \frac{(z^2-1)^2}{(z^2+1)^2},$$

and thus the theorem cannot be improved. Over the complex numbers, the Weierstrass $\wp$ function has four totally ramified values; that this is the most possible is a consequence of the Second Main Theorem.

Exercise 3.6.9. *Show that a non-Archimedean entire function f such that $f' \not\equiv 0$ can have at most one finite totally ramified value. What happens if the hypothesis $f' \not\equiv 0$ is dropped in positive characteristic?*

Theorem 3.6.10 (Adams & Straus). *Let f and g be two meromorphic functions on $\mathbf{F}$. If $\mathbf{F}$ has characteristic zero, assume f and g are not both constant. If $\mathbf{F}$ has positive characteristic, assume that neither $f' \equiv 0$ nor $g' \equiv 0$. Let $a_1, \ldots, a_4$ be distinct elements of $\mathbf{P}^1(\mathbf{F})$ and assume $f^{-1}(a_j) = g^{-1}(a_j)$ for $j = 1, \ldots, 4$. Then, $f = g$.*

Proof. We treat the case that $f' \not\equiv 0$ and $g' \not\equiv 0$. It isn't difficult to modify the proof to allow one of the functions to be constant. Without loss of generality, assume none of the a_j are infinity. It is easy to see that $T(f-g,\infty,r) \leq T(f,\infty,r) + T(g,\infty,r)$. By hypothesis,

$$N^{(1)}(f-g,0,r) \geq \sum_{j=1}^{4} N^{(1)}(f,a_j,r) = \sum_{j=1}^{4} N^{(1)}(g,a_j,r).$$

Applying the Second Main Theorem to both f and g we conclude that

$$\begin{aligned} 2T(f,\infty,r) + 2T(g,\infty,r) &\leq 2\sum_{j=1}^{4} N^{(1)}(f,a_j,r) - 2\log r + O(1) \\ &\leq 2N^{(1)}(f-g,0,r) - 2\log r + O(1) \\ &\leq 2T(f-g,0,r) - 2\log r + O(1) \\ &\leq 2T(f,\infty,r) + 2T(g,\infty,r) - 2\log r + O(1), \end{aligned}$$

which is a contradiction. $\square$

The example

$$f(z) = \frac{z}{z^2-z+1} \qquad \text{and} \qquad g(z) = \frac{z^2}{z^2-z+1}$$

shows that Theorem 3.6.10 is best possible since

$$f^{-1}(0) = g^{-1}(0), \qquad f^{-1}(1) = g^{-1}(1), \qquad \text{and} \qquad f^{-1}(\infty) = g^{-1}(\infty).$$

3.7 An's Defect Relation

Ru's observation that non-Archimedean inequalities of Second Main Theorem type follow from the First Main Theorem has been quite important. As an example, Ta Thi Hoài An [An] proved the following:

Theorem 3.7.1 (An's Defect Relation). *Let $X \subset \mathbf{P}^N$ be a projective variety and let $f : \mathbf{F} \to X$ be a non-constant non-Archimedean analytic map. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbf{P}^N$ in general position with X, and assume that the image of f is not completely contained in any of the D_j. Then,*

$$\sum_{j=1}^{q} \delta_f(D_j) \leq \dim X.$$

Remark. An proved Theorem 3.7.1 while she was visiting ICTP as a Junior Associate.

Here, in general position with X means that the intersection of any $\dim X + 1$ of the D_i and X is empty. Here the defects $\delta_f(D_j)$ measure f encountering the hypersurface D_i with less than expected frequency. In particular, a non-constant f cannot omit more than $\dim X$ of the D_i, unless it is completely contained in one of them. A defect relation such as this is unique to non-Archimedean analysis and has no counterpart in complex value distribution theory, in the sense that the dimension bounds the defect sum and that the bound is derived from the First Main Theorem. The deeper defect inequality of Eremenko and Sodin [ES] over the complex numbers takes a similar form to An's inequality, but with $\dim X$ replaced by $2 \dim X$, and it is true for entirely different reasons.

3.8 Concluding Remarks

The Second Main Theorem, with ramification, can also be proven for maps encountering hyperplanes in projective space; see [Bo 4], [KhTu], and [CY 1]. The techniques of this section can be used to prove that any non-Archimedean analytic map from the affine line $\mathbf{A}^1$ to an algebraic curve of positive genus must be constant; see [ChWa] for details. This was first proven by Berkovich [Brk] using his theory of analytic spaces. My lecture during the workshop in the third week will be about the degeneracy of images of non-Archimedean analytic maps to projective varities omitting divisors with sufficiently many components. Perhaps one of the most interesting things to investigate in non-Archimedean function theory is analogs of Big Picard theorems. For instance, one can prove [Ch 2] that a non-Archimedean analytic map from a punctured disc to an elliptic curve with good reduction must always extend across the puncture. But this need not be true for elliptic curves with bad reduction. That whether a Big Picard type theorem is true or not can depend on the reduction type of the target is a phenomenon completely foreign to the complex analytic situation.

4 Benedetto's Island Theorems

4.1 Ahlfors Theory of Covering Surfaces

In work that won him one of the first Fields Medals, Ahlfors [Ah 1] developed a theory of covering surfaces that both gave a geometric interpretation of Nevanlinna's Second Main Theorem as a generalization of the Gauss-Bonet Formula and extended it to the distribution of "domains," rather than "values," and also to classes of mappings more general than meromorphic functions, for instance quasiconformal mappings. The Ahlfors Five Islands Theorem was a consequence of his covering theory and has been an important tool in the study of complex dynamics; see [Brg 2]. Recall that as a consequence of the Second Main Theorem, a meromorphic function can have at most four totally ramified values. That means if f is a meromorphic function on $\mathbf{C}$ and $a_1, \ldots, a_5$ are five distinct values in $\mathbf{P}^1(\mathbf{C})$, then there must be a point z_0 in $\mathbf{C}$ such that $f(z_0)$ is one of the

a_j and $f(z_0) = a_j$ with multiplicity one. The Five Islands Theorem is the same statement, but with the values a_j replaced by domains.

Theorem 4.1.1 (Ahlfors Five Island Theorem). *Let f be a meromorphic function on $\mathbf{C}$ and let $D_1, \ldots, D_5$ be five simply connected domains in $\mathbf{P}^1(\mathbf{C})$ with disjoint closures. Then, there exists an open set U in $\mathbf{C}$ such that f is a conformal bijection between U and one of the domains $D_1, \ldots, D_5$.*

Remark. The theorem gets its name because Ahlfors thought of the five domains $D_1, \ldots, D_5$ as islands on the Riemann sphere. The theorem says that given five islands, a meromorphic function must cover at least one of the islands injectively.

In two significant papers, [Ben 1] and [Ben 2], R. L. Benedetto investigated non-Archimedean analogs of the Ahlfors island theorems. This lecture is intended as an introduction to Benedetto's work.

4.2 A Non-Archimedean Riemann Mapping Theorem?

The islands in Ahlfors's theorems are simply connected domains. What should the non-Archimedean analog be?

Proposition 4.2.1. *Let f be a non-constant analytic function on $\mathbf{B}_{\le r}$ and let $R = |f - f(0)|_r$. Then,*

$$f(\mathbf{B}_{\le r}) = \{w \in \mathbf{F} : |w - f(0)| \le R\}.$$

Proposition 4.2.1 says that the image of a disc under a non-constant non-Archimedean analytic function must be another disc. Thus, the non-Archimedean analog of the Riemann mapping theorem would be the trivial statement that given any bordered disc D in $\mathbf{F}$, there is an analytic map from D to $\mathbf{B}_{\le 1}$.

Proof. Let z in $\mathbf{B}_{\le r}$. Then,

$$|f(z) - f(0)| \le |f - f(0)|_r$$

by the Maximum Modulus Principle, and hence

$$f(\mathbf{B}_{\le r}) \subset \{w \in \mathbf{F} : |w - f(0)| \le R\}.$$

Now let w be such that $|w - f(0)| \le R$. Then,

$$|f - w|_r = |f - f(0) + f(0) - w|_r \le \max\{|f - f(0)|_r, |w - f(0)|\} = R.$$

Write

$$f(z) = f(0) + \sum_{k=1}^{\infty} c_k z^k, \qquad \text{and so} \qquad f(z) - w = f(0) - w + \sum_{k=1}^{\infty} c_k z^k.$$

By assumption,

$$\sup_{k \ge 1} |c_k| r^k = R, \qquad \text{and so} \qquad K(f - w, r) \ge 1.$$

Hence, $f(z) - w$ has a zero $\mathbf{B}_{\le r}$ by Theorem 2.4.4. □

Exercise 4.2.2. *If f is a non-constant analytic function on $\mathbf{B}_{<r}$, then $f(\mathbf{B}_{<r})$ is an unbordered disc, including the possibility that $f(\mathbf{B}_{<r}) = \mathbf{F}$, which can be viewed as an unbordered disc of infinite radius.*

By a bordered disc in $\mathbf{P}^1(\mathbf{F})$, we mean either a bordered disc in $\mathbf{F}$ or a set of the form

$$\{w \in \mathbf{F} : |w| \geq R > 0\} \cup \{\infty\}.$$

By an unbordered disc in $\mathbf{P}^1(\mathbf{F})$, we mean either an unbordered disc in $\mathbf{F}$ (including the possiblility of $\mathbf{F}$ itself), or a set of the form

$$\{w \in \mathbf{F} : |w| > R \geq 0\} \cup \{\infty\}.$$

Exercise 4.2.3. *If f is a non-constant meromorphic function on $\mathbf{B}_{\leq r}$ (resp. $\mathbf{B}_{<r}$), then $f(\mathbf{B}_{\leq r})$ (resp. $f(\mathbf{B}_{<r})$) is either all of $\mathbf{P}^1(\mathbf{F})$ or a bordered (resp. unbordered) disc in $\mathbf{P}^1(\mathbf{F})$.*

4.3 Non-Archimedean Analogs of the theorems of Bloch, Landau, Schottky, and Koebe

In [Ben 1], Benedetto formulated and proved non-Archimedean analogs of the classical theorems of Bloch, Landau, Schottky, and Koebe. These are stated here as exercises for the reader. The first is the most difficult and depends on the characteristics of $\mathbf{F}$ and $\widetilde{\mathbf{F}}$. See [Ben 1] for solutions to these exercises, as well as further commentary and references for the classical complex analogs.

Exercise 4.3.1 (Non-Archimedean Bloch's Constant).

(i) If char $\mathbf{F} = 0$, *let*

$$B = \begin{cases} 1 & \text{if char } \widetilde{\mathbf{F}} = 0 \\ |p|^{1/(p-1)} & \text{if char } \widetilde{\mathbf{F}} = p > 0. \end{cases}$$

Let f be analytic on $\mathbf{B}_{\leq 1}$ normalized so that $f(0) = 0$ and $f'(0) = 1$. Then, there is an unbordered disc U in $\mathbf{B}_{\leq 1}$ such that f is injective on U and such that $f(U)$ is an unbordered disc of radius B. Moreover, there exists an analytic function f on $\mathbf{B}_{\leq 1}$ such that $f(0) = 0$, $f'(0) = 1$, and such that if U is any bordered or unbordered disc in $\mathbf{B}_{\leq 1}$ on which f is injective, then $f(U)$ does not contain a bordered disc of radius B.

(ii) If char $\mathbf{F} > 0$, *then given $\varepsilon > 0$, there exists an analytic function f on $\mathbf{B}_{\leq 1}$ such that $f(0) = 0$, such that $f'(0) = 1$, and such that if U is any bordered or unbordered disc in $\mathbf{B}_{\leq 1}$ such that f is injective on U, then $f(U)$ does not contain a bordered disc of radius ε.*

Remark. The constant B in Exercise 4.3.1 (and zero if char $\mathbf{F} > 0$) can be called the non-Archimedean Bloch constant, and the examples of the exercise show the constant is sharp. Over the complex numbers, the existence of a positive constant B such that if f is holomorphic on the unit disc in $\mathbf{C}$ normalized so that $f'(0) = 1$, then f injectively covers some disc of radius B is a theorem of Bloch. The sharp value of B in the complex case is a long-standing conjecture, and remains unproven.

Exercise 4.3.2 (Non-Archimedean Landau's Constant). *If f is analytic on $\mathbf{B}_{\leq 1}$ normalized such that $f(0) = 0$ and $f'(0) = 1$, then $f(\mathbf{B}_{\leq 1}) \supseteq \mathbf{B}_{\leq 1}$.*

Remark. Exercise 4.3.2 says that the non-Archimedean Landau constant is 1, and the value 1 is clearly best possible. As with Bloch's constant, the sharp value of Landau's constant over the complex numbers is conjectured, but not yet proven.

Exercise 4.3.3 (Non-Archimedean Koebe 1/4-Theorem). *If f is analytic and injective on $\mathbf{B}_{\leq 1}$ and normalized so that $f(0) = 0$ and $f'(0) = 1$, then $f(\mathbf{B}_{\leq 1}) = \mathbf{B}_{\leq 1}$.*

Exercise 4.3.4 (Non-Archimedean Landau Theorem). *Let f be analytic and zero free on $\mathbf{B}_{\leq 1}$. Then $|f'(0)| < |f(0)|$.*

Exercise 4.3.5 (Non-Archimedean Schottky Theorem). *Let f be analytic and zero free on $\mathbf{B}_{\leq 1}$. Then, $|f(z)| = |f(0)|$ for all z in $\mathbf{B}_{\leq 1}$.*

4.4 Island Theorems

Entire Functions

Given Exercise 3.6.9, one might expect the following statement to be a non-Archimedean analog of the Ahlfors Island Theorem for entire functions.

Statement 4.4.1. *Let D_1 and D_2 be two disjoint unbordered discs in $\mathbf{F}$ and let f be an entire function on $\mathbf{F}$ such that $f' \not\equiv 0$. Then, there exists an unbordered disc U in $\mathbf{F}$ such that f is injective on U and such that $f(U) = D_1$ or $f(U) = D_2$.*

Unfortunately, as we will see in a moment, Statement 4.4.1 is false if char $\widetilde{\mathbf{F}} > 0$, even in the case that char $\mathbf{F} = 0$. We will also see that Statement 4.4.1 is true if char $\widetilde{\mathbf{F}} = 0$.

Before looking at some examples in positive characteristic, we give a general proposition.

Proposition 4.4.2. *Let f be analytic and injective on a bordered disc D of radius r containing the point a. Then, $f(D)$ is a bordered disc of radius at most $r|f'(a)|$.*

Proof. That $f(D)$ is a bordered disc is Proposition 4.2.1. Without loss of generality, assume f is given by a power series of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n.$$

Let $b > r|a_1|$. If b is in $f(\mathbf{B}_{\leq r})$, then

$$|a_1|r < |b| \leq |f|_r = \sup_{n \geq 1} |a_n| r^n,$$

and so $K(f, r) \geq 2$, and f is not injective on $\mathbf{B}_{\leq r}$. □

Difficulties in positive characteristic

We now explain a fundamental difference between the case of analytic functions over the complex numbers and non-Archimedean analytic functions when char $\widetilde{\mathbf{F}} > 0$.

Exercise 4.4.3. *Let G be a domain in $\mathbf{C}$ and let f be analytic on G. Let R be the set of ramification points of f, i.e.,*

$$R = \{z \in G : f'(z) = 0\}.$$

Let $B = f(R)$ be the set of branch points in $f(G)$. Let D be a simply connected domain in $f(G) \setminus B$. Then, there exists an analytic function g, called a branch of f^{-1}, on D with values in G such that $f(g(z)) \equiv z$ on D. This implies that the open set $U = g(D)$ has the property that f is injective on U and $f(U) = D$.

When char $\widetilde{\mathbf{F}} = 0$, the same property holds in the non-Archimedean case.

Exercise 4.4.4. *Let f be analytic on a disc $\mathbf{B}_{\leq r}$. Let $R \subset \mathbf{B}_{\leq r}$ be the set of ramification points and $B = f(R)$ be the set of branch points. Let D be a bordered or unbordered disc in $f(\mathbf{B}_{\leq r}) \setminus B$. Then, there exists an analytic function g on D with values in $\mathbf{B}_{\leq r}$ such that $f(g(z)) \equiv z$ on D.*

When char $\mathbf{F} = p > 0$, the statement in Exercise 4.4.4 is spectacularly false. The polynomial $f(z) = z + z^p$ is such that $f'(z) \equiv 1$, so that f has no ramification or branch points, but f fails to have an inverse function. The same phenomenon persists even when char $\mathbf{F} = 0 < p =$ char $\widetilde{\mathbf{F}}$. In this case, R is not empty, but rather

$$R = \{\zeta \in \mathbf{F} : \zeta^{p-1} = -1/p\},$$

and so $|\zeta| = |p|^{-1/(p-1)} > 1$ for each ζ in R. Hence, $|f(\zeta)| = |p|^{-p/(p-1)} > 1$, and so $\mathbf{B}_{\leq 1} \cap B = \emptyset$. Nonetheless, f is not injective on any disc mapping onto $\mathbf{B}_{\leq 1}$. [Exercise: prove this.]

An Example When char $\mathbf{F} > 0$.

Consider the case that char $\mathbf{F} = \text{char}\, \widetilde{\mathbf{F}} = p > 0$. Clearly the hypothesis $f' \not\equiv 0$ in Statement 4.4.1 is necessary because for pure p-th powers, such as $f(z) = z^p$, every value is totally ramified and f is nowhere injective. However, Statement 4.4.1 remains false even with this hypothesis.

Example 4.4.5 *(*[Ben 1, pp. 598]*).* Let char $\mathbf{F} = p > 0$, let $\varepsilon > 0$, and let c be an element of $\mathbf{F}$ such that $|c| > \varepsilon^{-(p-1)}$. Then, $f(z) = z + cz^p$ is not injective on any bordered disc of radius ε (characteristic p!) and hence f does not injectively cover any bordered disc of radius ε by Proposition 4.4.2, since $f'(z) \equiv 1$.

Example 4.4.5 not only shows that Statement 4.4.1 is false, but it also shows that no island theorem can be true for all entire functions (or even polynomials) when char $\mathbf{F} > 0$, even if the number of islands is increased or one requires the islands to be very small.

An Example When char $\mathbf{F} = 0 < p = \text{char}\, \widetilde{\mathbf{F}}$.

We now show that Statement 4.4.1 is false when char $\widetilde{\mathbf{F}} = p > 0 = \text{char}\, \mathbf{F}$, even if one increases the number of islands or adds an additional restriction that the islands be "small."

Example 4.4.6. Let char $\widetilde{\mathbf{F}} = p > 0 = \text{char}\, \mathbf{F}$. Let a_i be infinitely many points in $\mathbf{B}_{\le 1}$ such that $|a_i| = 1$ for all i and such that $|a_i - a_j| = 1$ for all $i \neq j$, which is possible since $\widetilde{\mathbf{F}}$ is algebraically closed, and hence infinite. Let $1 \ge \varepsilon > 0$ and choose n such that $|p|^n < \varepsilon$. Let D_i be unbordered discs containing a_i of radius ε. Let $f(z) = z^{p^n}$. Then, f does not injectively cover any of the D_i, which are disjoint since $|a_i - a_j| = 1$. Indeed, for each i, let ξ_i be a point in $\mathbf{F}$ such that $f(\xi_i) = a_i$, and note that $|\xi_i| = 1$. Let U be a bordered disc containing ξ_i on which f is injective. Clearly, f is not injective on $\mathbf{B}_{\le 1}$, and so the radius of U is at most 1. It then follows from Proposition 4.4.2 that $f(U)$ is a disc containing a_i of radius at most $|p|^n < \varepsilon$, and therefore D_i is not injectively covered by f.

Benedetto's Island Theorem for Analytic Functions on a Disc

As we have seen, Statement 4.4.1 is false when char $\widetilde{\mathbf{F}} > 0$. I introduced this lecture with the statements of Ahlfors's island theorems for functions meromorphic or holomorphic on $\mathbf{C}$. In fact, Ahlfors's theorems apply to functions meromorphic or holomorphic on a disc that satisfy certain additional hypotheses. The Ahlfors island theorems for functions meromorphic or analytic on $\mathbf{C}$ then follow by showing that if f is a non-constant meromorphic or analytic function on $\mathbf{C}$, then when f is restricted to sufficiently large discs, it satisfies the additional hypotheses of the associated disc island theorem. Benedetto's point of view is that although the island theorem for non-Archimedean entire functions is not true in general, there is a good non-Archimedean analog of Ahlfors's island theorem for functions analytic on discs.

Before stating the theorem, we introduce some convenient notation. For a non-Archimedean analytic function, define

$$f^{\#}(z) = \begin{cases} \dfrac{|f'(z)|}{\max\{1, |f(z)|\}} & \text{if } f(z) \neq \infty \\ \left|\left(\dfrac{1}{f}\right)'(z)\right| & \text{if } f(z) = \infty. \end{cases}$$

The quantity $f^{\#}$ is a non-Archimedean analog of the spherical derivative in complex analysis, and it is also convenient to adopt the notation

$$||f^{\#}||_r = \frac{|f'|_r}{\max\{1, |f|_r\}}.$$

One sees that $f^{\#}$ is the natural measure of the distortion of f considered as a map to $\mathbf{P}^1(\mathbf{F})$.

Theorem 4.4.7 (Benedetto's Analytic Island Theorem). *Let D_1 and D_2 be two disjoint unbordered discs in $\mathbf{F}$, each of finite radius. There exist explicit constants C_1 and C_2 depending only on D_1, D_2 and the characteristics of $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ with $C_2 = 0$ when $\operatorname{char} \widetilde{\mathbf{F}} = 0$, such that the following holds. Given f analytic on $\mathbf{B}_{<1}$ with $f^{\#}(0) > C_1$ and $r||f^{\#}||_r \geq C_2$ for some $0 < r < 1$, then there exists an unbordered disc U in $\mathbf{B}_{<1}$ such that f is injective on U and such that $f(U) = D_1$ or $f(U) = D_2$.*

Remark. The hypothesis $f^{\#}(0) > C_1$ is a natural necessary hypothesis for an analytic function on a disc. This hypothesis ensures that both D_1 and D_2 are in the image of f, and clearly without some hypothesis to ensure that the islands are in the image of f, no such theorem would be possible. The hypothesis $r||f^{\#}||_r \geq C_2$ for some r is automatically satisfied when $\operatorname{char} \widetilde{\mathbf{F}} = 0$ and has the effect of ruling out the positive characteristic pathologies that we explored above. Benedetto's point of view is that this second hypothesis is in some sense a natural non-Archimedean analog to the hypothesis in Ahlfors's island theorem for meromorphic functions on a disc that the mean covering number be sufficiently big with respect to the relative boundary length, two notions from Ahlfors's theory of covering surfaces that I will not attempt to make precise here. A significant difference, though, between the complex and non-Archimedean cases is that in the complex case, if f is a non-constant meromorphic function on $\mathbf{C}$, then $f(rz)$ will satisfy Ahlfors's hypothesis for all r sufficiently large, and therefore result in his island theorem for non-constant meromorphic functions on $\mathbf{C}$, whereas in the non-Arcimedean setting when $\operatorname{char} \widetilde{\mathbf{F}} > 0$, we have seen examples of functions f such that none of the functions $f_a = f(az)$, no matter how large $|a|$ is, satisfy the hypothesis $r||f_a^{\#}||_r \geq C_2$ for some $0 < r < 1$.

I will not give the idea of the proof of Theorem 4.4.7 in the most interesting case when $\operatorname{char} \widetilde{\mathbf{F}} > 0$ in these lectures and instead simply refer the reader to Benedetto's paper [Ben 1]. I will instead prove a special case of the theorem in the case that $\operatorname{char} \widetilde{\mathbf{F}} = 0$ that explains the essential steps in this most simple of cases.

Proposition 4.4.8. *Assume* $\operatorname{char} \widetilde{\mathbf{F}} = 0$. *Let*

$$D_0 = \{z \in \mathbf{F} : |z| < 1\} \qquad \textit{and} \qquad D_1 = \{z \in \mathbf{F} : |z-1| < 1\}.$$

Let f be analytic on $\mathbf{B}_{\leq 1}$ such that $f^{\#}(0) \geq 1$. Then, there exists an unbordered disc U in $\mathbf{B}_{<1}$ such that f is injective on U and such that $f(U) = D_0$ or $f(U) = D_1$.

Proof. Write f as a power series,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

The hypothesis $|f'(0)| \geq \max\{1, |f(0)|\}$ exactly says that $|a_1| \geq |a_0 - b|$ for all $|b| \leq 1$. In particular, $K(f-b, 1) \geq 1$ for all $|b| \leq 1$, and hence

$$f(\mathbf{B}_{\leq 1}) \supseteq \mathbf{B}_{\leq 1}.$$

Note that I have just provided the solution to Exercise 4.3.2, and so far the characteristic zero hypothesis has not been used.

I now explain what the hypothesis $\operatorname{char} \widetilde{\mathbf{F}} = 0$ provides. Let $K = K(f, 1) = K(f-1, 1) \geq 1$. By definition, for $j > K$, we have $|a_j| < |a_K|$. The characteristic zero hypothesis then implies

$$|j||a_j| < |K||a_K| \quad \text{for } j > K \qquad \text{and} \qquad |K||a_K| \geq |j||a_j| \quad \text{for } 1 \leq j \leq K,$$

since $|j| = |K| = 1$. Hence,

$$K(f', 0, 1) = K - 1. \tag{18}$$

For a subset $X \subseteq \mathbf{B}_{\leq 1}$, we define the following counting functions. We let $n(f,\{0,1\},X)$ denote the number of points z in X such that $f(z)$ is in $\{0,1\}$, and we count with points repeated according to multiplicity. Similarly, we let $n(f',0,X)$ denote the number of zeros of f' in X, again repeated according to multiplicity. The key fact we need to show in order to prove the proposition is that there exists a point z_0 in $\mathbf{B}_{\leq 1}$ with $f(z_0) \in \{0,1\}$ and such that if U is any disc containing z_0, then

$$n(f,\{0,1\},U) > 2n(f',0,U). \tag{19}$$

Since $K(f,1) = K(f-1,1) = K$, by Theorem 2.4.3, there are $2K$ points z, counting multiplicity, in $\mathbf{B}_{\leq 1}$ such that $f(z) \in \{0,1\}$. Suppose that (19) is false. Then, we may choose disjoint discs X_i containing each of the finitely many points z_j such that $f(z_j) \in \{0,1\}$. Note that we are not claiming that each X_i contains exactly one z_j; the same disc X_i may contain several of the z_j. Nonetheless, by the disjointness of the X_i, we would have

$$2K = n(f,\{0,1\},\mathbf{B}_{\leq 1}) = \sum_i n(f,\{0,1\},X_i) \leq 2\sum_i n(f',0,X_i) \leq 2n(f',0,\mathbf{B}_{\leq 1}) = 2(K-1),$$

by (18), which would be a contradiction. Thus, (19) is established. for some point z_0. The inequality in (19) implies that $f'(z_0) \neq 0$. Without loss of generality, assume $f(z_0) = 0$, for otherwise we may replace f with $1-f$. Also, by a linear change of coordinate, we may assume without loss of generality that $z_0 = 0$. Now let $R < 1$ be the smallest radius such that $|f|_R = 1$. I claim that if $U = \mathbf{B}_{<R}$, then f is injective on U and $f(U) = D_0$. That $f(U) = D_0$ follows immediately from $|f|_R = 1$, since if $|b| < 1$, we can find $r < R$ with $|f|_r = |b|$. Let $r < R$. Because $|f|_r < 1$, we have that

$$n(f,\{0,1\},\mathbf{B}_{\leq r}) = K(f,r).$$

Also,

$$n(f',0,\mathbf{B}_{\leq r}) = K(f',r) = K(f,r) - 1,$$

where again the last inequality follows from the hypothesis that char $\widetilde{\mathbf{F}} = 0$ in the same way as (18). By (19),

$$2(K(f,r)-1) = 2n(f',0,\mathbf{B}_{\leq r}) < n(f,\{0,1\},\mathbf{B}_{\leq r}) = K(f,r),$$

and hence $K(f,r) < 2$. Thus, f is injective on $\mathbf{B}_{\leq r}$, and hence also on U. $\square$

Benedetto's Island Theorem for Meromorphic Functions on a Disc

I will conclude this lecture with the statement of Benedetto's Four Island Theorem for meromorphic functions on a disc. When char $\widetilde{\mathbf{F}} > 0$, there is another added complication for meromorphic functions beyond what was already present for analytic functions. Thus, it is more complicated to formulate a hypothesis to exclude the full range of pathologies one can meet in positive characteristic. Benedetto states his hypothesis using the language of Berkovich analytic spaces. We will let $\mathcal{P}^1$ be the Berkovich analytic space associated to $\mathbf{P}^1$ and we will let $\mathcal{B}_{<1}$ be the Berkovich space associated to $\mathbf{B}_{<1}$. Then a meromorphic function f on $\mathbf{B}_{<1}$ naturally extends to a mapping from $\mathcal{B}_{<1}$ to $\mathcal{P}^1$. Also, recall that each ν in $\mathcal{B}_{<1}$ is naturally associated to a multiplicative semi-norm on the ring of analytic functions on $\mathbf{B}_{<1}$, and so therefore extends to a semi-norm $||\ ||_\nu$ on the field of meromorphic functions on $\mathbf{B}_{<1}$. Recall also that to each point ν in $\mathbf{B}_{<1}$, one can associate an embedded family of bordered discs

$$B(a_i,r_i) = \{z \in \mathbf{F} : |z-a_i| \leq r_i\} \subset B(a_{i-1},r_{i-1}) = \{z \in \mathbf{F} : |z-a_{i-1}| \leq r_{i-1}\} \subset \mathbf{B}_{<1},$$

and hence one naturally associates to ν a "radius," $r(\nu)$ defined by $r(\nu) = \inf r_i$. Although the family of discs $B(a_i,r_i)$ is not uniquely determined by ν, the quantity $r(\nu)$ is. Finally, for ν with $r(\nu) > 0$ in $\mathcal{B}_{<1}$ and f meromorphic on $\mathbf{B}_{<1}$, we let $||f^\#||_\nu$ denote

$$||f^\#||_\nu = \frac{||f'||_\nu}{\max\{1, ||f||_\nu\}}.$$

We are then finally able to state Benedetto's theorem.

Theorem 4.4.9 (Benedetto's Meromorphic Island Theorem). *Let $D_1, \ldots, D_4$ be four disjoint unbordered discs in $\mathbf{P}^1$. Let μ be a point of $\mathcal{P}^1$ such that no connected component (in the Berkovich topology) of $\mathcal{P}^1 \setminus \{\mu\}$ intersects more than two of the D_i. There exist explicit constants C_1 and C_2 depending only on the D_i, on μ, and on the characteristics of $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ with $C_2 = 0$ when $\operatorname{char} \widetilde{\mathbf{F}} = 0$, such that the following holds. Given f meromorphic on $\mathbf{B}_{<1}$ with $f^{\#}(0) > C_1$ and $r(\nu)||f^{\#}||_\nu \geq C_2$ for all points ν in $\mathcal{B}_{<1}$ such that $f(\nu) = \mu$, then there exists an unbordered disc U in $\mathbf{B}_{<1}$ such that f is injective on U and such that $f(U) = D_i$ for some i from 1 to 4.*

I simply refer the reader to [Ben 2] for a discussion of the proof, but we conclude by mentioning one example.

Example 4.4.10 *(*[Ben 2, Ex. 6.2]*).* Examples of the form

$$\left(z + \frac{c}{z^{p^n}}\right) \prod_{i=1}^{N} \left(1 + \frac{c}{(z - a_i)^{p^n}}\right),$$

where $|a_i - a_j| = 1$ if $i \neq j$ and $|a_i| = 1$ for all i show that even if one allows very many very small islands when $\operatorname{char} \widetilde{\mathbf{F}} = p$, one cannot significantly weaken the hypotheses of Theorem 4.4.9 as one can show that such examples satisfy the inequality involving $r(\nu)||f^{\#}||_\nu$ for some ν with $f(\nu) = \mu$, but not for all ν with $f(\nu) = \mu$, and for these functions, the conclusion of the theorem is false for small islands around the a_i.

References

[Adms] W. W. Adams, Transcendental Numbers in the P-Adic Domain, *American J. Math.* **88** (1966), 279–308.

[AdSt] W. W. Adams and E. G. Straus, Non-Archimedian analytic functions taking the same values at the same points, *Illinois J. Math.* **15** (1971), 418–424.

[Ah 1] L. V. Ahlfors, Zur Theorie der Überlagerungsfachen, *Acta Math.* **65** (1935), 157–194.

[Ah 2] L. V. Ahlfors, *Complex Analysis,* Third Edition, McGraw-Hill, 1979.

[Am] Y. Amice, *Les nombres p-adiques,* Presse Universitaires de France, 1975.

[An] Ta Thi Hoài An, A defect relation for non-Archimedean analytic curves in arbitrary projective varieties, *Proc. Amer. Math. Soc.* **135** (2007), 1255–1261.

[Ben 1] R. L. Benedetto, Non-Archimedean holomorphic maps and the Ahlfors Islands theorem, *Amer. J. Math.* **125** (2003), 581–622.

[Ben 2] R. L. Benedetto, An Ahlfors islands theorem for non-Archimedean meromorphic functions. *Trans. Amer. Math. Soc.* **360** (2008), 4099–4124.

[Brg 1] W. Bergweiler, Iteration of meromorphic functions, *Bull. Amer. Math. Soc. (N.S.)* **29** (1993), 151–188.

[Brg 2] W. Bergweiler, The role of the Ahlfors five islands theorem in complex dynamics, *Conform. Geom. Dyn.* **4** (2000), 22–34.

[Brk] V. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields,* Mathematical Surveys and Monographs **33**, American Mathematical Society, 1990.

[Bo 1] A. Boutabaa, Theorie de Nevanlinna p-Adique, *Manuscripta Math.* **67** (1990), 251-269.

[Bo 2] A. Boutabaa, *Sur la théorie de Nevanlinna p-adique,* Thése de Doctorat, Université Paris 7, 1991.

[Bo 3] A. Boutabaa, Applications de la theorie de Nevanlinna p-adique, *Collect. Math.* **42** (1991), 75–93.

[Bo 4] A. Boutabaa, Sur les courbes holomorphes p-adiques, *Annales de la Faculté des Sciences de Toulouse* **V** (1996), 29–52.

[BoEs] A. Boutabaa and A. Escassut, Nevanlinna theory in positive characteristic and applications, *Ital. J. Pure Appl. Math.* **23** (2008), 45–66.

[Ch 1] W. Cherry, Non-Archimedean analytic curves in Abelian varieties, *Math. Ann.* **300** (1994), 393-404.

[Ch 2] W. Cherry, Non-Archimedean big Picard theorems, unpublished manuscript, arXiv:math/0207081v1.

[ChTo] W. Cherry and C. Toropu, Generalized ABC theorems for non-Archimedean entire functions of several variables in arbitrary characteristic, *Acta Arith.* **136** (2009), 351–384.

[ChWa] W. Cherry and J. T.-Y. Wang, “Non-archimedean analytic maps to algebraic curves” in W. Cherry and C.-C. Yang (eds.), *Value Distribution Theory and Complex Dynamics,* Contemporary Math. **303**, American Mathematical Society, 2002, pp. 7–35.

[ChRu] W. Cherry and M. Ru, Rigid Analytic Picard Theorems, *Amer. J. Math.* **126** (2004), 873–889.

[CY 1] W. Cherry and Z. Ye, Non-Archimedean Nevanlinna theory in several variables and the non-Archimedean Nevanlinna inverse problem, *Trans. Amer. Math. Soc.* **349** (1997), 5043–5071.

[CY 2] W. Cherry and Z. Ye, *Nevanlinna's theory of value distribution. The second main theorem and its error terms,* Springer-Verlag, 2001.

[Co 1] C. Corrales-Rodrigáñez, *Nevanlinna Theory in the p-Adic Plane,* Ph.D. Thesis, University of Michigan, 1986.

[Co 2] C. Corrales-Rodrigáñez, Nevanlinna Theory on the p-Adic Plane, *Annales Polonici Mathematici* **LVII** (1992), 135-147.

[Dwk] B. Dwork, On exponents of p-adic differential modules, *J. Reine Angew. Math.* **484** (1997), 85–126.

[ES] A. È. Erëmenko and M. L. Sodin, Distribution of values of meromorphic functions and meromorphic curves from the standpoint of potential theory. (Russian) *Algebra i Analiz* **3** (1991), 131–164; English translation in *St. Petersburg Math. J.* **3** (1992), 109–136.

[Har] T. Harase, MR1437300 (98h:12008) in *Mathematical Reviews,* American Mathematical Society, 1998.

[Hay] W. K. Hayman, *Meromorphic Functions,* Oxford Mathematical Monographs, Clarendon Press, Oxford 1964.

[HY 1] P.-C. Hu and C. C. Yang, *Meromorphic functions over non-Archimedean fields,* Mathematics and its Applications **522**, Kluwer Academic Publishers, 2000

[HY 2] P.-C. Hu and C.-C. Yang, The "*abc*" conjecture over function fields, *Proc. Japan Acad. Ser. A Math. Sci.* **76** (2000), 118–120.

[HY 3] P.-C. Hu and C.-C. Yang, A note on the abc conjecture, *Comm. Pure Appl. Math.* **55** (2002), 1089–1103.

[HY 4] P.-C. Hu and C.-C. Yang, A note on Browkin-Brzeziński conjecture in *Ultrametric Functional Analysis,* Contemp. Math. **384**, American Mathematical Sococciety, Providence, RI, 2005, 101–109.

[Kh 1] Hà Huy Khoái, On p-adic meromorphic functions, *Duke Math. J.* **50** (1983), 695–711.

[Kh 2] Hà Huy Khoái, La hauteur des fonctions holomorphes p-adiques de plusieurs variables, *C. R. Acad. Sci. Paris Sér. I Math.* **312** (1991), 751-754.

[KhQu] Hà Huy Khoái and My Vinh Quang, "On p-adic Nevanlinna Theory," in *Complex analysis, Joensuu 1987,* Lecture Notes in Mathematics **1351**, Springer-Verlag 1988, 146-158.

[KhTu] Hà Huy Khoái and Mai Van Tu, p-Adic Nevanlinna-Cartan Theorem, *Internat. J. Math.* **6** (1995), 719-731.

[Lai] I. Laine, *Nevanlinna theory and complex differential equations,* de Gruyter Studies in Mathematics **15**, Walter de Gruyter & Co., 1993.

[Lang] S. Lang, *Algebra,* Third Edition, Addison-Wesley, 1993.

[Laz] M. Lazard, Les zéros des fonctions analytiques d'une variable sur un corps valué complet, *Inst. Hautes Études Sci. Publ. Math.* **14** (1962), 47–75.

[Nev] R. Nevanlinna, *Analytic functions,* Die Grundlehren der mathematischen Wissenschaften **162**, Springer-Verlag, 1970.

[Rob] A. Robert, *A Course in p-adic Analysis,* Graduate Texts in Mathematics **198**, Springer-Verlag, 2000.

[Ru] M. Ru, A note on p-adic Nevanlinna theory, *Proc. Amer. Math. Soc.* **129** (2001), 1263–1269.

[Schn] L. G. Shnirel'man, O funkcijah v normirovannyh algebraičeski zamknutyh telah [On functions in normed algebraically closed divison rings], *Izvestija AN SSSR* **2** (1938), 487–498.

[vdP] M. van der Put, Essential singularities of rigid analytic functions, *Nederl. Akad. Wetensch. Indag. Math.* **43** (1981), 423–429.