TOPICS IN p-ADIC FUNCTION THEORY

WILLIAM CHERRY

1. Picard Theorems

I would like to begin by recalling the Fundamental Theorem of Algebra.

Theorem 1.1. (Fundamental Theorem of Algebra) *A non-constant polynomial of one complex variable takes on every complex value. Moreover, if the polynomial is of degree d, then every complex value is taken on d times, counting multiplicity.*

Because entire functions have power series expansions, they are sort of like polynomials of infinite degree. Picard's well-known theorem is a complex analytic analog of the Fundamental Theorem of Algebra.

Theorem 1.2. (Picard's (Little) Theorem) *A non-constant entire function takes on all but at most one complex value. Moreover, a transcendental entire function must take on all but at most one complex value infinitely often.*

The function e^z shows that a complex entire function can indeed omit one value.

Lately, it has become fashionable to prove p-adic versions of value distribution theorems, of which Picard's Theorem is an example, though not a recent one. More recent examples can be found in the works listed in the references section. Recall that the p-adic absolute value $| \;|_p$ on the rational number field $\mathbf{Q}$ is defined as follows. If $x \in \mathbf{Q}$ is written $p^k a/b$, where p is a prime, k is an integer, and a and b are integers relatively prime to p, then $|x|_p = p^{-k}$. Completing $\mathbf{Q}$ with respect to this absolute value results in the field of p-adic numbers, denoted $\mathbf{Q}_p$. Taking the algebraic closure of $\mathbf{Q}_p$, extending $| \;|_p$ to it, and then completing once more results in a complete algebraically closed field, denoted $\mathbf{C}_p$, and often referred to as the p-adic complex numbers.

Recall that the absolute value $| \;|_p$ satisfies a very strong form of the triangle inequality, namely $|x + y|_p \le \max\{|x|_p, |y|_p\}$. This is referred to as a non-Archimedean triangle inequality, and this non-Archimedean triangle inequality is what accounts for most of the differences between function theory on $\mathbf{C}_p$ and on $\mathbf{C}$.

Recall that an infinite series $\sum a_n$ converges under a non-Archimedean norm if and only if $\lim\limits_{n\to\infty} a_n = 0$. By an entire function on $\mathbf{C}_p$, one means a formal power series $\sum\limits_{n=0}^{\infty} a_n z^n$, where a_n are elements of $\mathbf{C}_p$, and $\lim\limits_{n\to\infty} |a_n|_p r^n = 0$, for every $r > 0$, so that plugging in any element of $\mathbf{C}_p$ for z results in an absolutely convergent series.

Most of what I will discuss here is true over an arbitrary algebraically closed field complete with respect to a non-Archimedean absolute value, but for simplicity's sake, I will stick with the concrete case $\mathbf{C}_p$ here.

If one tries to prove Picard's Theorem for p-adic entire functions, what one gets is the following theorem.

Theorem 1.3. **(p-Adic Case)** *A non-constant p-adic entire function must take on every value in $\mathbf{C}_p$. Moreover, a transcendental p-adic entire function must take on every value in $\mathbf{C}_p$ infinitely often.*

Proof. Let $f(z) = \sum a_n z^n$ be a p-adic entire function, so $\lim_{n\to\infty} |a_n|_p r^n = 0$, for all $r > 0$. Denote by $|f|_r = \sup |a_n|_p r^n$. The graph of

$$\log r \mapsto \log |f|_r = \sup_{n\geq 0}\{\log |a_n|_p + n \log r\}$$

is piecewise linear and closely related to what's known as the Newton polygon. In particular, ther zeros of f occur at the "corners" of the graph of $\log r \mapsto \log |f|_r$ (*c.f.,* [Am] and [BGR]).

For r close to zero, $|f|_r = |a_0|_p$, provided $a_0 \neq 0$. Moreover, it is clear that if f is not constant, then for all r sufficiently large, $|f|_r \neq |a_0|_p$. Hence, the graph of $\log r \mapsto \log |f|_r$ has a corner, and hence f has a zero.

If f is transcendental, then f has infinitely many non-zero Taylor coefficients, and thus for every n, there exists r_n such that for all $r \geq r_n$, we have $|f|_r > |a_n|_p r^n$. Hence, $\log r \mapsto \log |f|_r$ must have infinitely many corners, and so f has infinitely many zeros. □

Note that Theorem 1.3 is an even closer analogy to the Fundamental Theorem of Algebra than Picard's Theorem was, since p-adic entire functions, like polynomials, cannot omit any values. Thus, in this respect, the function theory of p-adic entire functions is more closely related to the function theory of polynomials than it is to the function theory of complex holomorphic functions. That will be the theme of this survey.

2. Algebraic Curves

My second illustration that p-adic function theory is more like that of polynomials comes from considering Riemann surfaces. Let X be a projective algebraic curve of genus g. Then, the three analogous theorems we have are:

Theorem 2.1. **(Polynomial Case)** *If $f\colon \mathbf{C} \to X$ is a non-constant polynomial mapping, then $g = 0$.*

Theorem 2.2. **(Complex Case)** *If $f\colon \mathbf{C} \to X$ is a non-constant holomorphic mapping, then $g \leq 1$.*

Theorem 2.3. **(p-Adic Case)** *If $f\colon \mathbf{C}_p \to X$ is a non-constant p-adic analytic mapping, then $g = 0$.*

The polynomial case follows from the Riemann-Hurwitz formula, which says that the genus of the image curve cannot be greater than the genus of the domain.

The complex case was again proved by Picard. Riemann surfaces of genus ≥ 2 have holomorphic universal covering maps from the unit disc, and thus any holomorphic map form $\mathbf{C}$ to a Riemann surface of genus ≥ 2 lifts to a holomorphic map to the unit disc, which must then be constant by Liouville's Theorem.

The p-Adic analog of this theorem was proven only recently, by V. Berkovich [Ber].

One of the major difficulties in p-adic function theory is the fact that the natural p-adic topology is totally disconnected, and therefore analytic continuation in these circumstances is a delicate task. Moreover, geometric techniques that are commonplace in complex analysis cannot be applied in the p-adic case. In order to prove his p-adic analog of Picard's Theorem, Berkovich developed a theory of p-adic analytic spaces that enlarges the natural p-adic spaces so that they become nice topological spaces, and geometric techniques, such as universal covering spaces, can be used to prove theorems.

3. Berkovich Theory

Berkovich's theory is somewhat deep, and I do not have ther required space to go into it in much detail here. However, the reader may find the following brief description of his theory helpful. The interested reader is encouraged to look at: [Ber], [Ber 2], and [BGR]. The last reference covers the more traditional theory of rigid analytic spaces.

Although one can associate a Berkovich space to any p-adic analytic variety, we will concentrate here on the special case of the unit ball in $\mathbf{C}_p$, which is the local model for smooth p-adic analytic spaces, at least in dimension one.

Consider the closed unit ball $\mathbf{B} = \{z \in \mathbf{C}_p : |z|_p \le 1\}$. The p-adic analytic functions on $\mathbf{B}$ are of the form $\sum a_n z^n$, with $\lim\limits_{n\to\infty} |a_n|_p = 0$. These functions form a Banach algebra $\mathcal{A}$ under the norm $|f|_{0,1} = \sup\limits_n |a_n|_p$.

The Berkovich space associated to $\mathbf{B}$ consists of all bounded multiplicative semi-norms on $\mathcal{A}$. This space is provided with the weakest topology such that all maps of the form $|\ | \mapsto |f|$, $f \in \mathcal{A}$ are continuous maps to the real numbers with their usual topology. Here $|\ |$ denotes one of the bounded multiplicative semi-norms in the Berkovich space.

Berkovich spaces have many nice topological properties, such as local compactness and local arc-connectedness. They also have universal covering spaces, which are again Berkovich spaces.

For $f \in \mathcal{A}$, $z_0 \in \mathbf{B}$, and $0 \le r \le 1$, define $|f|_{z_0,r}$ by $|f|_{z_0,r} = \sup_n |c_n|_p r^n$, where $f = \sum c_n (z - z_0)^n$, or in other words, the c_n are the coefficients of the Taylor expansion of f about z_0. Note that if $r = 0$, then $|f|_{z_0,0} = |f(z_0)|_p$, and note that by the non-Archimedean triangle inequality, if $|z_0 - w_0|_p \le r$, then $|\ |_{z_0,r} = |\ |_{w_0,r}$. There are in fact more bounded multiplicative semi-norms on $\mathbf{B}$ than these, but these are the main ones to think about.

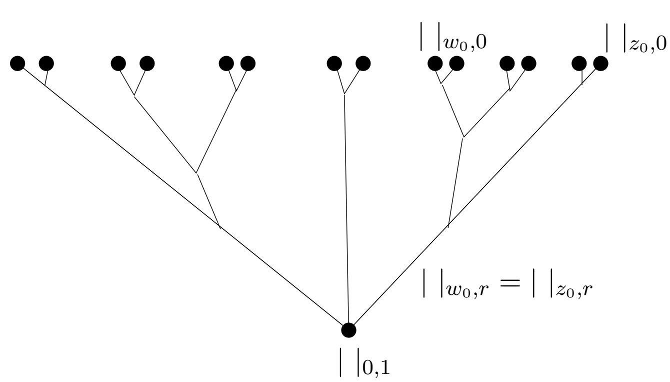

Figure 1.

Figure 1 gives a sort of intuitive "tree-like" representation for the Berkovich space associated to $\mathbf{B}$. The dots at the top correspond to the totally disconnected points in $\mathbf{B}$. Of course there are infinitely many of these, and there are points arbitrarily close together, much like a Cantor set. The lines represent the connected continuum of additional multiplicative semi-norms connecting the Berkovich space. There are of course infinitely many places where lines join together, and the junctures are by no means discrete. Finally, the point at the bottom corresponds to the one semi-norm $|\ |_{z_0,1}$ which is the same for all points z_0 in $\mathbf{B}$.

We say that two points z_0 and w_0 in $\mathbf{B}$ are in the same residue class if $|z_0 - w_0|_p < 1$. This leads to a concept called "reduction," whereby the space is "reduced" to the space of residue classes. The reduction of $\mathbf{B}$ can be naturally identified with $\mathbf{A}^1_{\mathbf{F}_p^{\mathrm{alg}}}$, the affine line over the algebraic closure of the field of p elements. This process of reduction extends to the Berkovich space associated to $\mathbf{B}$, and there is a reduction mapping π from the Berkovich space $\mathbf{B}$ to $\mathbf{A}^1_{\mathbf{F}_p^{\mathrm{alg}}}$. The reduction mapping π has what I would call an anti-continuity property, in that π^{-1} of a Zariski open sets in $\mathbf{A}^1_{\mathbf{F}_p^{\mathrm{alg}}}$ will be closed in the Berkovich topology and π^{-1} of a Zariski closed set will be open in the Berkovich topology.

In Figure 1, two points in the Berkovich tree are in different residue classes if their branches do not join except at the one point $|\ |_{0,1}$, which is kind of like a "generic" point in algebraic geometry, and is in fact the inverse image of the generic point in $\mathbf{A}^1_{\mathbf{F}_p^{\mathrm{alg}}}$ under the reduction map. Thus, three residue classes are shown in Figure 1.

4. Abelian Varieties

In my Ph.D. thesis [Ch 1], I extended Berkovich's Theorem to Abelian varieties. See also: [Ch 2] and [Ch 3].

Theorem 4.1. (Cherry) *If $f\colon \mathbf{C_p} \to A$ is a p-adic analytic map to an Abelian variety, then f must be constant.*

Proof sketch.

$$\begin{tikzcd}
1 \arrow[r] & T \arrow[r] & G \arrow[r] \arrow[d] & B \arrow[r] \arrow[dr, "\pi_B"] & 1 \\
& \mathbf{C}_p \arrow[r, "f"] \arrow[ur, dashed, "f^!"] & A & & \widetilde{B}
\end{tikzcd}$$

T is a product of multiplicative groups (*i.e.* a multiplicative torus).
G is the universal cover of A in the sense of Berkovich, and a semi-Abelian variety.
B is an Abelian variety with good reduction, meaning it has a reduction mapping π_B to an Abelian variety $\widetilde{B}$ over $\mathbf{F}_p^{\mathrm{alg}}$.

Figure 2.

Step 1. First, we use Berkovich theory to lift f to a map $f^!\colon \mathbf{C}_p \to G$ to the universal covering of A.

Step 2. Next we use p-adic uniformization ([BL 1], [BL 2], [DM]) to identify G as a semi-Abelian variety, as in Figure 2.

Step 3. Then, we use reduction techniques. We get a map

$$\mathbf{C}_p \to G \to B \to \widetilde{B}.$$

This map must be constant because if it were not we would induce a non-constant rational map from the projective line over $\mathbf{F}_p^{\mathrm{alg}}$ to the Abelian variety $\widetilde{B}$. Thus, the image in B lies above a single smooth point in $\widetilde{B}$. The inverse image of a smooth point in $\widetilde{B}$ is isomorphic to an open ball in $\mathbf{C}_p^n$, where n is the dimension of B. Thus, the map to B is also constant, by the p-adic version of Liouville's Theorem, for example.

Step 4. Thus, we only need consider mappings from $\mathbf{C}_p$ to T. But,

$$T \cong \mathbf{C}_p \setminus \{0\} \times \cdots \times \mathbf{C}_p \setminus \{0\}.$$

The projection onto each factor is constant by the p-Adic version of Picard's Little Theorem. $\square$

Because p-adic analytic maps to Abelian varieties must be constant, the following conjecture seems plausible.

Conjecture 4.2. *Let X be a smooth projective variety. If there exists a non-constant p-adic analytic map from $\mathbf{C}_p$ to X, then there exists a non-constant rational mapping from $\mathbf{P}^1$ to X.*

5. Value Sharing

One of the more striking consequences of Nevanlinna theory is Nevanlinna's theorem that if two non-constant meromorphic functions f and g share five values, then f must equal g, [Ne]. The polynomial version of this was taken up by Adams and Straus in [AS].

Theorem 5.1. (Adams and Straus) *If f and g are two non-constant polynomials over an algebraically closed field of characteristic zero such that $f^{-1}(0) = g^{-1}(0)$ and $f^{-1}(1) = g^{-1}(1)$, then $f \equiv g$.*

Proof. Assume $\deg f \geq \deg g$ and consider $[f'(f-g)]/[f(f-1)]$. This is a polynomial because if $f(z) = 0$ or 1, then $f(z) = g(z)$ by assumption, and hence the zeros in the denominator are canceled by the zeros in the numerator, and the f' in the numerator takes care of multiple zeros. On the other hand, the degree of the numerator is strictly less than the degree of the denominator, so the numerator must be identically zero. In other words f is constant, or f is identically equal to g. $\square$

Theorem 5.2. (Adams and Straus) *If f and g are non-constant p-adic (characteristic zero) analytic functions such that $f^{-1}(0) = g^{-1}(0)$, and $f^{-1}(1) = g^{-1}(1)$, then $f \equiv g$.*

Proof. We may assume without loss of generality that there exist $r_j \to \infty$ such that $|f|_{r_j} \geq |g|_{r_j}$. Let $h = [f'(f-g)]/[f(f-1)]$. Then, h is entire since, as in the polynomial case, zeros in the denominator are always matched by zeros in the numerator. On the other hand, by the non-Archimedean triangle inequality, we have for r_j sufficiently large that

$$|h|_{r_j} = \left|\frac{f'}{f}\right|_{r_j} \cdot \frac{|f-g|_{r_j}}{|f-1|_{r_j}} \leq \left|\frac{f'}{f}\right|_{r_j} \cdot \frac{|f|_{r_j}}{|f|_{r_j}} = \left|\frac{f'}{f}\right|_{r_j}.$$

Now, I claim $|f'/f|_r \le r^{-1}$, and therefore $|h|_{r_j} \to 0$ as $r_j \to \infty$. Hence, $h \equiv 0$, and again, either f is constant of $f \equiv g$. $\square$

The claim that $|f'/f|_r \le 1/r$ is the p-adic form of the Logarithmic Derivative Lemma, and note this is much stronger than what is true in the complex case.

Theorem 5.3. **(p-Adic Logarithmic Derivative Lemma)** *If f is a p-Adic analytic function, then $|f'/f|_r \le 1/r$.*

Proof. Write $f = \sum a_n z^n$. Then, since $|n|_p \le 1$, we have

$$|f'|_r = \sup_{n\ge1}\{|na_n|_p r^{n-1}\} = \frac{1}{r}\sup_{n\ge1}\{|na_n|_p r^n\} \le \frac{1}{r}\sup_{n\ge0}\{|a_n|_p r^n\} = \frac{1}{r}|f|_r \quad \square$$

Notice the similarity in both the proof and the statement of both of Adams and Straus's theorems.

An active topic of current research has to do with so called "unique range sets." Rather than considering functions which share distinct values, one considers finite sets and functions f and g such that $f^{-1}(S) = g^{-1}(S)$. Here, Boutabaa, Escassut, and Haddad [BEH] gave a nice characterization for unique range sets of polynomials, in the counting multiplicity case.

Theorem 5.4. **(Boutabaa, Escassut, and Haddad)** *If f and g are polynomials over an algebraically closed field F of characteristic zero, and if S is a finite subset of F such that $f^{-1}(S) = g^{-1}(S)$, counting multiplicity, then either $f \equiv g$ or there exist constants A and B, $A \ne 0$, such that $g = Af + B$ and $S = AS + B$.*

Proof. Let $S = \{s_1, \dots, s_n\}$ and let

$$P(X) = (X - s_1)\cdots(X - s_n).$$

Then, $P(f)$ and $P(g)$ are polynomials with the same zeros, counting multiplicity by the assumption $f^{-1}(S) = g^{-1}(S)$. Thus, $P(f)/P(g)$ is some non-zero constant C, and if we set $F(X,Y) = P(X) - CP(Y)$, we have $F(f,g) = 0$. Thus, $z \mapsto (f(z), g(z))$ is a rational component of the possibly reducible algebraic curve $F(X,Y) = 0$. Because $F(X,Y) = 0$ has n distinct smooth points at infinity in $\mathbf{P}^2$ (characteristic zero!) and because $(f(z), g(z))$ has only one point at infinity, $(f(z), g(z))$ must in fact be a linear component of $F(X,Y) = 0$. $\square$

Boutabaa, Escassut, and Haddad also made a preliminary analysis of the p-adic entire analog of their theorem, and solved the case when the cardinality of S equals three completely. C.-C. Yang and I, [CYa], combined Berkovich's Picard theorem with their argument to complete the p-adic entire case.

Theorem 5.5. **(Cherry and Yang)** *If f and g are p-adic entire functions and S is a finite subset of $\mathbf{C}_p$ such that $f^{-1}(S) = g^{-1}(S)$, counting multiplicity, then there exist constants A and B, with $A \ne 0$, such that $g = Af + B$, and $S = AS + B$.*

Proof. Again, set

$$P(X) = (X - s_1)\cdots(X - s_n).$$

Again, $P(f)/P(g)$ is a constant $C \ne 0$. Again, set $F(X,Y) = F(X) - CF(Y)$. By Berkovich's p-Adic Picard Theorem, $(f(z), g(z))$ is contained in a rational component of $F(X,Y) = 0$. Thus, there exist rational functions u and v, and a p-adic entire function h, such that $f = u(h)$ and $g = v(h)$. It is then easy to see that u and v must in fact be polynomials, and we are then back to the polynomial case, thinking of h as a variable. $\square$

6. Concluding Remarks

In many respect, it appears that algebraic geometry, rather than complex Nevanlinna theory, is the appropriate model for p-adic value distribution theory. At least, that is what I hope this survey has conveyed to the reader. This leads me to a general principle.

Principle 6.1. *Appropriately stated theorems about the value distribution of polynomials should also be true for p-adic entire functions. Similarly, theorems for rational functions should also be true for p-adic meromorphic functions.*

Conjecture 4.2 is a special case of this principle. With some luck, solving a p-adic problem based on the above principle might help us better understand complex Nevanlinna theory. For example, it would be reasonable to make the following conjecture.

Conjecture 6.2. *If $f\colon \mathbf{C}_p \to X$ is a p-adic analytic map to a $K3$ surface X, the the image of f must be contained in a rational curve.*

This conjecture can be thought of as a special case of a p-adic version of the Green-Griffiths conjecture [GG] that says a holomorphic curve in a smooth projective variety of general type must be algebraically degenerate. One might hope to attack Conjecture 6.2 since much is known about $K3$ surfaces and they have a close connection to Abelian varieties. It might also be that finding a proof for Conjecture 6.2 would shed some light on an attack of the general Green-Griffiths conjecture over the complex numbers.

References

[AS] W. Adams and E. Straus, *Non-Archimedian analytic functions taking the same values at the same points,* Illinois J. Math. **15** (1971), 418–424.

[Am] Y. Amice, *Les nombres p-adiques,* Presses Universitaires de France, 1975.

[Ber] V. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields,* AMS Surveys and Mographs **33**, 1990.

[Ber 2] V. Berkovich, *Etale cohomology for non-Archimedean analytic spaces,* Inst. Hautes Etudes Sci. Publ. Math. **78** (1993), 5–161.

[BGR] S. Bosch, U. Güntzer and R. Remmert, *Non-Archimedean Analysis,* Springer-Verlag, 1984.

[BL 1] S. Bosch and W. Lütkebohmert, *Stable Reduction and Uniformization of Abelian Varieties* I, Math. Ann. **270** (1985), 349–379.

[BL 2] S. Bosch and W. Lütkebohmert, *Stable Reduction and Uniformization of Abelian Varieties* II, Invent. Math. **78** (1984), 257–297.

[Bo 1] A. Boutabaa, *Theorie de Nevanlinna p-Adique,* Manuscripta Math. **67** (1990), 251–269.

[Bo 2] A. Boutabaa, *Sur la théorie de Nevanlinna p-adique,* Thése de Doctorat, Université Paris 7, 1991.

[Bo 3] A. Boutabaa, *Applications de la theorie de Nevanlinna p-adique,* Collect. Math. **42** (1991), 75–93.

[Bo 4] A. Boutabaa, *Sur les courbes holomorphes p-adiques,* Annales de la Faculté des Sciences de Toulouse **V** (1996), 29–52.

[BEH] A. Boutabaa, A. Escassut, and L. Haddad, *On uniqueness of p-adic entire functions,* Indag. Math. (N.S.) **8** (1997), 145–155.

[Ch 1] W. Cherry, *Hyperbolic p-Adic Analytic Spaces,* Ph.D. Thesis, Yale University, 1993.

[Ch 2] W. Cherry, *Non-Archimedean analytic curves in Abelian varieties,* Math. Ann. **300** (1994), 393–404.

[Ch 3] W. Cherry, *A non-Archimedean analogue of the Kobayashi semi-distance and its non-degeneracy on abelian varieties,* Illinois J. Math. **40** (1996), 123–140.

[CYa] W. Cherry and C.-C. Yang, *Uniqueness of non-Archimedean entire functions sharing sets of values counting multiplicity,* Proc. Amer. Math. Soc., to appear.

[CYe] W. Cherry and Z. Ye, *Non-Archimedean Nevanlinna theory in several variables and the non-Archimedean Nevanlinna inverse problem,* Trans. Amer. Math. Soc. **349** (1997), 5043–5071.

[Co 1] C. Corrales-Rodrigáñez, *Nevanlinna Theory in the p-Adic Plane,* Ph.D. Thesis, University of Michigan, 1986.

[Co 2] C. Corrales-Rodrigáñez, *Nevanlinna Theory on the p-Adic Plane,* Annales Polonici Mathematici **LVII** (1992), 135–147.

[DM] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus,* Inst. Hautes Etudes Sci. Publ. Math. No. **36** (1969), 75–109.

[GG] M. Green and P. Griffiths, *Two applications of algebraic geometry to entire holomorphic mappings,* The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), Springer-Verlag 1980, 41–74.

[Hà 1] Hà Huy Khoái, *On p-Adic Meromorphic Functions,* Duke Math. J. **50** (1983), 695–711.

[Hà 2] Hà Huy Khoái, *La hauteur des fonctions holomorphes p-adiques de plusieurs variables,* C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), 751–754.

[HMa] Hà Huy Khoái and Mai Van Tu, *p-Adic Nevanlinna-Cartan Theorem,* Internat. J. Math. **6** (1995), 719–731.

[HMy] Hà Huy Khoái and My Vinh Quang, *On p-adic Nevanlinna Theory,* in Lecture Notes in Mathematics **1351**, Springer-Verlag 1988, 146–158.

[Ne] R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes,* Paris, 1929.

Department of Mathematics, P.O. Box 305118, University of North Texas, Denton, TX 76203-5118, USA

E-mail address: wcherry@unt.edu