Lectures on Holomorphic Curves in Abelian Varieties
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June 15–19, 2009

Lecture 1: Introduction and Green's Theorem on Holomorphic Curves in Complex Tori

Lecture 2: Nevanlinna's Lemma on the Logarithmic Derivative, Jet Bundles, and Bloch's Theorem on Holomorphic Curves in Abelian Varieties

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Draft In Progress: Last Updated: June 19, 2009

*It is my pleasure to acknowledge the generous financial support provided by Academia Sinica and National Central University which made these lectures possible. These lectures are largely based on Min Ru's concise exposition of the subject in [Ru].

1 Introduction and Green's Theorem on Holomorphic Curves in Complex Tori

1.1 Abelian Varieties

In addition to having had a standard introductory graduate course in complex analysis, I will assume that the audience is familiar with the basic theory of Abelian varieties; in fact, I will not use much of that theory, and I will recall in the next few paragraphs most of what we will need. I also assume the reader is familiar with the language of contemporary algberaic geometry. The following are monographs giving detailed treatments of Abelian varieties: [Mi], [Lng 1], [Mu]. Hartshorne [Har] is the standard reference for contemporary algebraic geometry. Some standard introductory graduate textbooks on complex analysis include: [Ah], [Co], [Lng 2], [No 2].

An **Abelian variety** is a complete group variety. It turns out that all Abelian varieties are projective and commutative; I will recall below the proof that they are commutative. Abelian varieties over the complex numbers $\mathbf{C}$ can always be realized as $\mathbf{C}^n/\Lambda$, where Λ is a lattice of rank $2n$. When $n > 1$, it is not true that $\mathbf{C}^n/\Lambda$ for an arbitrary lattice of rank $2n$ need be an Abelian variety because they may in fact not be algebraic varieties. Such objects are called **complex tori**, or perhaps **complete complex tori**, to distinguish them from products of the multiplicative group $\mathbf{G}_m(\mathbf{C}) = \mathbf{C}^\times = \mathbf{C} \setminus \{0\}$, which are also called "tori." The main results to be described in these lectures are about complex analysis, so many also hold for an arbitrary complex torus, whether it is an Abelian variety or not. Nonetheless, I will restrict attention to the case of Abelian varieties when it is simpler (or more algebraic) to do so, even when the results hold for more general complex tori. A closed subvariety B of an Abelian variety A is called an **Abelian subvariety** if B is also a subgroup of A. A closed analytic subvariety S of a complex torus T is called a **sub-torus** if it is a subgroup of T.

I begin with a useful proposition about rational or analytic maps from complete spaces, which Milne [Mi, Th. 1.1] refers to as "rigidity."

Proposition 1.1. *Let X, Y, and Z be algebraic (resp. complex analytic) varieties such that X is complete and $X \times Y$ is geometrically irreducible. Let $f : X \times Y \to Z$ be a morphism such that there exist (closed) points x_0 in X, y_0 in Y, and z_0 in Z with the property that*

$$f(\{x_0\} \times Y) = f(X \times \{y_0\}) = \{z_0\}.$$

Then, $f(X \times Y) = \{z_0\}$.

Proof. Let U be an affine open neighborhood of z_0 and denote by $q : X \times Y \to Y$ the projection onto the second factor. The set $Z \setminus U$ is Zariski closed in Z, whereby $f^{-1}(Z \setminus U)$ is Zariski closed in $X \times Y$. Hence

$$W = \{y \in Y : f(x, y) \notin U \text{ for some } x \in X\} = q(f^{-1}(Z \setminus U))$$

is Zariski closed in Y since X is complete. Note that in the algebraic case, the fact that q is a closed map is essentially the definition of "complete". Now, $V = Y \setminus W$ is Zariski open in Y and is not empty since $y_0 \in V$. Also, $f(X \times V) \subset U$. If y is in V, then

$$f : X \times \{y\} \to U$$

is constant since $X \times \{y\} \cong X$ is complete and U is affine. On the other hand, $(x_0, y) \in X \times \{y\}$, and so

$$f(X \times \{y\}) = f(x_0, y) = z_0$$

and hence $f(X \times V) = \{z_0\}$. Since $X \times Y$ is irreducible, $X \times V$ is dense in $X \times Y$, and so $f(X \times Y) = \{z_0\}$. □

Corollary 1.2. *Any morphism between Abelian varieties (resp. complete complex analytic group varieties) is a translation composed with a group homomorphism.*

Proof. Let $\phi : A \to B$ be a morphism between Abelian varieties (resp. complete complex analytic group varietes). Although we do not yet know that the group operation is commuttive, we write it additively. Let 0 denote the identity element of A and let $\psi : A \to B$ denote the morphism $\psi(a) = \phi(a) - \phi(0)$. Let $\Psi : A \times A \to B$ be the morphism defined by

$$\Psi(a_1, a_2) = \psi(a_1 + a_2) - \psi(a_1) - \psi(a_2).$$

Then,

$$\Psi(0, a_2) = \Psi(a_1, 0) = 0,$$

and so $\Psi(a_1, a_2) \equiv 0$ by the proposition. In other words, ψ is a group homomorphism, and $\phi(a) = \psi(a) + \phi(0)$ is the group homomorphism ψ composed with translation in B by $\phi(0)$. $\square$

Corollary 1.3. *Abelian varieties (resp. complete complex analytic group varieties) are commutative.*

Proof. Let A be an Abelian variety (resp. a complete complex analytic group variety). Let $\phi : A \to A$ be the morphism defined by $\phi(a) = -a$. As $\phi(0) = 0$, we know ϕ is a group homomorphism by the previous corollary, but this exactly means A is commutative. $\square$

In the algebraic category, rigidity continues to hold even without completeness when the target is an Abelian variety.

Lemma 1.4. *Let V be a complete non-singular irreducible algebraic variety, let U be Zariski open in V and let $\phi : U \to A$ be a morphism from U to an Abelian variety A. Then ϕ extends to a morphism $\phi : V \to A$ from V to A.*

Proof. As this proof takes us too far from the ideas I want to stress in these lectures, the proof is omitted here. See [Mi, §I.3] or [Lng 1, §II.1]. $\square$

Corollary 1.5. *Let $\phi : G \to A$ be a morphism (or more generally a rational map) from an algebraic group variety G to an Abelian variety A. Then ϕ is the composition of a group homomorphism with a translation.*

Proof. Once one shows that G can be completed to a non-singular variety, the proof is the same as the proof of Corollary 1.2 after applying Lemma 1.4. I omit the details in general, but point out that obviously the additive group $\mathbf{G}_a$ and the multiplicative group $\mathbf{G}_m$ can be completed to the projective line $\mathbf{P}^1$. $\square$

Corollary 1.6. *Any morphism from the projective line $\mathbf{P}^1$ to an Abelian variety A must be constant.*

Proof. Let $\phi : \mathbf{P}^1 \to A$ be a morphism. By the previous corollary, we know that

$$\phi(x + y) = \phi(x) + \phi(y) - \phi(0) \qquad \text{and} \qquad \phi(xy) = \phi(x) + \phi(y) - \phi(1)$$

for all x and y in $\mathbf{G}_m = \mathbf{P}^1 \setminus \{0, \infty\}$. Hence, $\phi(x + y) - \phi(xy)$ is constant on $\mathbf{G}_m \times \mathbf{G}_m$. The morphism $\Phi : \mathbf{A}^1 \times \mathbf{A}^1 \to A$ defined by $\Phi(x, y) = \phi(x + y) - \phi(xy)$ is constant on the dense open subset $\mathbf{G}_m \times \mathbf{G}_m \subset \mathbf{A}^1 \times \mathbf{A}^1$, and hence constant everywhere. Setting $y = 0$, we see that ϕ must be constant. $\square$

The key idea of the above proof is that the map from $\mathbf{P}^1$ to A cannot be too far from a group homomorphism. It ends up being constant because dense open subsets of $\mathbf{P}^1$ carry two different group structures. To foreshadow another theme of these lectures, let me now give another proof of Corollary 1.6, this one valid only over the complex numbers, that makes use of the fact that an Abelian variety over the complex numbers is a complex torus.

Proposition 1.7. *Any holomorphic map from the projective line* $\mathbf{P}^1$ *to a complete complex torus* T *must be constant.*

Proof. Let $f : \mathbf{P}^1 \to T$ be holomorphic. Let $p : \mathbf{C}^n \to \mathbf{C}^n/\Lambda = T$ be the quotient map defining T as a complex torus, and note that p is also a universal covering map. Hence, f lifts to a holomorphic map $\tilde{f} : \mathbf{P}^1 \to \mathbf{C}^n$, such that $f = p \circ \tilde{f}$. Now, $\tilde{f}$ must be constant (by the maximum modulus principle) since $\mathbf{P}^1$ is compact. Hence, f is also constant. □

1.2 Green's Theorem on Holomorphic Curves in Complex Tori

Up to now in considering maps $f : X \to A$, the completeness of X has been essential. The subject of these notes is holomorphic curves. A **holomorphic curve** in a complex space X is a holomorphic map $f : \mathbf{C} \to X$. Of course, $\mathbf{C}$ is not complete and not every holomorphic map f from $\mathbf{C}$ need extend to a holomorphic map from $\mathbf{P}^1$. In fact, when X is a projective algebraic variety, those maps which do extend to $\mathbf{P}^1$ are exactly the algebraic maps. Thus, in studying holomorphic curves, we will not be able to use completeness in the same way. Not every holomorphic curve in an Abelian variety is a translation composed with a group homomorphism, but we will nonetheless be able to show that every holomorphic curve in an Abelian variety is not "too far" from a translated group homomorphism. As an introduction to this idea I present a theorem of M. Green [Gr]:

Theorem 1.8 (Green). *Let* X *be a closed analytic subvariety of a complete complex torus* T*. If there exists a non-constant holomorphic curve in* X*, then there exists a positive dimensional subtorus* $S \subset T$ *such that some translate of* S *is contained in* X*.*

I begin by recalling Möbius automorphisms of discs. Let $\mathbf{D}(r) = \{z \in \mathbf{C} : |z| < r\}$.

Exercise 1.9. *Let* a *be a point in* $\mathbf{D}(r)$*, and define*

$$\phi(z) = \frac{z+a}{1+\dfrac{\bar{a}z}{r^2}}.$$

Then, ϕ *is an automorphism of* $\mathbf{D}(r)$ *and*

$$\frac{|\phi'(z)|}{1-\dfrac{|\phi(z)|^2}{r^2}} = \frac{1}{1-\dfrac{|z|^2}{r^2}}.$$

Remark. The expression

$$\frac{|dz|}{1-\dfrac{|z|^2}{r^2}}$$

defines a Hermetian metric on $\mathbf{D}(r)$ called the **hyperbolic** metric. Exercise 1.9 then has the geometric interpretation that the automorphisms ϕ are isometries for the hyperbolic metric.

I now state a general lemma about closures of subgroups.

Lemma 1.10. *Let* G *be a group with a topology such that translation by an arbitrary element of* G *is a homeomorphism of* G *and such that the map* $x \mapsto x^{-1}$ *is a homeomorphism of* G*. Then, the closure of an abstract subgroup of* G *is again a subgroup of* G*.*

Remark. For our application, it is important not to assume G is a topological group, since we will not necessarily have that the multiplication map from $G \times G$ to G is continuous with respect to the product topology on $G \times G$; the Zariski topology on $G \times G$ is not the product topology on $G \times G$.

Proof. Let H be an abstract subgroup of G. Then

$$\overline{H}^{-1} = \{x^{-1} : x \in \overline{H}\}$$

is a closed subset containing $H^{-1} = H$, and hence $\overline{H} \subset \overline{H}^{-1}$. Applying the inverse map to this inclusion yields $\overline{H}^{-1} \subset \overline{H}$, and so $\overline{H}^{-1} = \overline{H}$.

It remains to check that $\overline{H}$ is closed under multiplication. Let $h \in H$. Then, $hH \in H \subset \overline{H}$, and so $H \subset h^{-1}\overline{H}$. The set $h^{-1}\overline{H}$ is closed since translation is a homeomrophism, and hence $\overline{H} \subset h^{-1}\overline{H}$, or in other words $h\overline{H} \subset \overline{H}$. Thus $H\overline{H} \subset \overline{H}$ and similarly $\overline{H}H \subset \overline{H}$. Finally, let $h \in \overline{H}$. Then, $\overline{H}H \subset \overline{H}$ exactly means that $hH \subset \overline{H}$. Therefore $H \subset h^{-1}\overline{H}$, and so $\overline{H} \subset h^{-1}\overline{H}$ since $h^{-1}\overline{H}$ is closed. In other words, $h\overline{H} \subset \overline{H}$, which means $\overline{H}\,\overline{H} \subset \overline{H}$. □

Proof of Theorem 1.8. Let $f : \mathbf{C} \to X \subset T = \mathbf{C}^n/\Lambda$ be a non-constant holomorphic map. Let $p : \mathbf{C}^n \to T$ be the universal covering from $\mathbf{C}^n$ onto T. Lift f to a map $\tilde{f} : \mathbf{C} \to \mathbf{C}^n$ such that $p \circ \tilde{f} = f$. The map $\tilde{f}$ is a vector

$$\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n),$$

and by $|\tilde{f}'|$, I mean

$$|\tilde{f}'| = \sqrt{|\tilde{f}_1'|^2 + \cdots + |\tilde{f}_n'|^2}.$$

If $|\tilde{f}'|$ is bounded, then $\tilde{f}$ is linear by Lioville's theorem. Hence $\tilde{f}$ is the translate of a group homomorphism from $\mathbf{C}$ to $\mathbf{C}^n$. As p is also a group homomorphism, f is a translate of a group homomorphism, and so the image of f is the translate of a subgroup of T contained in X. By Lemma 1.10, the Zariski closure of the image of f, which is contained in X since X is closed, is the translate of a closed subgroup of T, hence the translate of a subtorus of T contained in X.

If $|\tilde{f}'|$ is not bounded, then proceed as follows. Let $r_k \to \infty$. Choose a_k with $|a_k| \leq r_k$ such that

$$|\tilde{f}'(z)|\left(1 - \frac{|z|^2}{r_k^2}\right) \leq |\tilde{f}'(a_k)|\left(1 - \frac{|a_k|^2}{r_k^2}\right)$$

for all $|z| \leq r_k$. Let

$$\phi_k(z) = \frac{z + a_k}{1 + \dfrac{\bar{a}_k z}{r_k^2}},$$

and let $g_k = \tilde{f}_k \circ \phi_k$. Then, by Exercise 1.9 and our choice of a_k, we have

$$|g_k'(z)| = |\tilde{f}' \circ \phi_k(z)||\phi_k'(z)| = |\tilde{f}' \circ \phi_k(z)|\frac{1 - \dfrac{|\phi_k(z)|^2}{r_k^2}}{1 - \dfrac{|z|^2}{r_k^2}} \leq \frac{|\tilde{f}'(a_k)|\left(1 - \dfrac{|a_k|^2}{r_k^2}\right)}{1 - \dfrac{|z|^2}{r_k^2}} \leq \frac{4}{3}|g_k'(0)| \qquad (*)$$

for all $|z| \leq r_k/2$. Fix a compact subset K of $\mathbf{C}^n$ such that $p(K) = T$. Choose λ_k in Λ such that $g_k(0) - \lambda_k$ is in K. If $|g_k'(0)| \leq 1$, then let $h_k(z) = g_k(z) - \lambda_k$. Otherwise, let $t = |g_k'(0)|^{-1}$ and let $h_k(z) = g_k(tz) - \lambda_k$. Then, for k sufficiently large since $|\tilde{f}'|$ is unbounded, we have $|h_k'(0)| = 1$. We also have $|h_k'(z)| \leq 4/3$ for $|z| \leq r_k/2$ by $(*)$. Notice that by the derivative bound, the h_k form an equicontinous family with $h_k(0)$ bounded and the image of h_k contained in the image of $\tilde{f}$. Hence, by the Ascoli-Arzela theorem, we can find a subsequence h_{k_j} which converges locally uniformly to a holomorphic map $h : \mathbf{C} \to \mathbf{C}^n$. The map h has the property that $|h'(0)| = \lim |h_{k_j}'(0)| = 1$ and $|h'(z)| \leq 4/3$ for all z in $\mathbf{C}$. Because $|h'(0)| = 1$, the map h is non-constant, and because $|h'|$ is bounded, again, by Liouville's theorem, h is linear, and hence the translate of a group homomorphism. In additon, the image of h is contained in the closure of the image of $\tilde{f}$. Thus, the Zariski closure of the image of $p \circ h$ in T is the translate of a non-trivial subtorus of T contained in X. □

The idea of reparametrizing by the Möbius automorphisms of a disc as in the above proof has been used in complex function theory at least as far back as Landau [Lnd, pp. 618–619]. Lohwater and Pommerenke [LoPo] combined this with rescaling and taking limits in studying normal functions. Zalcman [Za 1] observed the proof applies to normal families and has written a survey illustrating its importance [Za 2]. In the context used here, this idea goes by the name of Brody's Reparmetrization Lemma [Br], or see [Lng 3] or [Ko]. Rickman has formulated a rescaling lemma that is particluary useful for quantitative estimates; see *e.g.* [ChEr].

As useful as reparametrization and rescaling is, the fact that the limiting map does not necessarily lie in the image of f but possibly only its closure does limit how far one can take this technique in studying the images of arbitrary holomorphic curves in varieties. A holomorphic curve whose derivative remains bouned with respect to a Hermetian metric is called a **Brody curve**, and recent work on Winkelmann [Wi] shows that the possible images of Brody curves in a variety can be very different than the images of arbitrary holomorphic curves.

2 Nevanlinna's Lemma on the Logarithmic Derivative, Jet Bundles, and Bloch's Theorem on Holomorphic Curves in Abelian Varieties

In the last lecture we saw that if $f : \mathbf{C} \to A$ is a non-constant holomorphic curve in an Abelian variety, then the Zariski closure of the image of f must contain the translate of an Abelian subvariety of A. In this lecture, we will prove the more precise statement that the Zariski closure of the image of f is itself the translate of an Abelian subvariety.

2.1 Nevanlinna's Theory of Value Distribution

In the 1920's, R. Nevanlinna introduced important new tools to the study of meromorphic functions that enabled complex function theory to significantly advance. I briefly introduce his theory here.

Let f be a meromorphic function on $\mathbf{C}$. Nevanlinna introduced the following functions associated to f that concern the behaviour of the function f on $\mathbf{D}(r)$, the disc of radius r. Nevanlinna's theory then examines how the three functions can be related as $r \to \infty$.

Given a point a in $\mathbf{P}^1$, the **unintegrated counting function function** $n(f, a, r)$ is defined to be the number of times the function f takes on the value a in the closed disc $|z| \le r$, counting multiplicity. The **integrated counting function** $N(f, a, r)$ is then defined by

$$N(f, a, r) = \int_0^r [n(f, a, t) - n(f, a, 0)] \frac{dt}{t} + n(f, a, 0) \log r.$$

If we let $\mathrm{ord}_z(f)$ denote the order of vanishing of f at z, with negative orders indicating poles and if let $\mathrm{ord}_z^+(f) = \max\{\mathrm{ord}_z(f), 0\}$, then

$$N(f, a, r) = \sum_{0<|z|\le r} \mathrm{ord}_z^+(f - a) \log \frac{r}{|z|} + \mathrm{ord}_0^+(f - a) \log r.$$

It may seem that $n(f, a, r)$ is the more natural quantity, but we will see shortly why the logarithmic average $N(f, a, r)$ is more convenient to work with; we observe already that one advantage $N(f, a, r)$ has over $n(f, a, r)$ is that $N(f, a, r)$ is continuous as a function of r.

Again given a point a in $\mathbf{P}^1$, the **mean proximity function** $m(f, a, r)$ is defined by

$$m(f, a, r) = \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|} \frac{d\theta}{2\pi},$$

where if $a = \infty$, one should replace $\log^+ |f(re^{i\theta}) - a|^{-1}$ with $\log^+ |f(re^{i\theta})|$ and $\log^+$ is defined by $\max\{\log, 0\}$. The mean proximity function measures, on average, how close f is to the value a on the circle $|z| = r$.

The Nevanlinna **characteristic** function or **height** of f is defined by

$$T(f, a, r) = m(f, a, r) + N(f, a, r).$$

Nevanlinna referred to the function T as a "characteristic" function. It enjoys functorial properties analogous to those satisfied by height functions in arithmetic geometry, and thus number theorists tend to prefer to refer to it as a "height."

I now recall the Jensen formula from introductory complex analysis:

Theorem 2.1 (Jensen Formula). *Let f be meromorphic and not identicall zero in a neighborbood of the closed disc $|z| \le r$. Then,*

$$\log|\mathrm{ilc}(f,0)| = \int_0^{2\pi} \log|f(re^{i\theta})| \frac{d\theta}{2\pi} - \sum_{0<|z|\le r} \mathrm{ord}_z(f) \log\frac{r}{|z|} - \mathrm{ord}_0(f)\log r.$$

Here ilc(f, 0) denotes the "initial Laurent coefficient" for f at 0. In other words, expand f in a Laurent series about 0 as

$$f(z) = \sum_{n=n_0}^{\infty} c_n z^n$$

with $c_{n_0} \neq 0$ and define $\mathrm{ilc}(f,0) = c_{n_0}$. Because

$$\log x = \log^+ x - \log^+ \frac{1}{x} \qquad \text{and} \qquad \mathrm{ord}_z(f) = \mathrm{ord}_z^+(f) - \mathrm{ord}_z^+\left(\frac{1}{f}\right),$$

we see that the Jensen formula can be rewritten as

$$\log|\mathrm{ilc}(f,0)| = m(f,\infty,r) - m(f,0,r) + N(f,\infty,r) - N(f,0,r) = T(f,\infty,r) - T(f,0,r),$$

or in other words

$$T(f,0,r) = T(f,\infty,r) + O(1).$$

Moreover, it is easy to see that for a in $\mathbf{C}$,

$$m(f-a,\infty,r) = m(f,\infty,r) + O(1) \qquad \text{and} \qquad N(f-a,\infty,r) = N(f,\infty,r).$$

Thus, applying Jensen to $f - a$, we see

$$T(f,a,r) = T(f-a,0,r) = T(f-a,\infty,r) + O(1) = T(f,\infty,r) + O(1),$$

and so we have

Theorem 2.2 (Nevanlinna's First Main Theorem). $T(f,a,r) = T(f,\infty,r) + O(1)$.

As the First Main Theorem tells ust that $T(f,a,r)$ is, up to a bounded term, independent of a, I will tend to write just $T(f,r)$ to emphasize this. For the sake of definiteness, I will take $T(f,r)$ to be defined to be $T(f,\infty,r)$.

Recalling that the characteristic or height function is the sum of the counting and proximity functions, we get the following interpretations. Since for all values a, the sums $m(f,a,r)+N(f,a,r)$ are, up to a bounded term, independent of a, we see that by combining how often the function f attains the value a with how often f remains near the function a, we get a quantity that essentially does not depend on a. Thus, for instance, the function e^z never takes on the value 0, but to compensate for this, it must remain "near" the value 0 on a large percentage of the circle

$|z| = r$. The First Main Theorem can also be interpreted as giving an upper bound on $N(f, a, r)$ in terms of something essentially independent of a. In this sense, the First Main Theorem is analogous to the part of the Fundamental Theorem of Algebra that says a polynomial of degree d takes on the value a at most d times.

Although I will not make use of it in these lectures, for the sake of completeness I also state Nevanlinna's Second Main Theorem, which is the analog in function theory to the statement that a polynomial of degree d takes on every complex value at least d times, counting multiplicity.

Theorem 2.3 (Nevanlinna's Second Main Theorem). *Let f be a non-constant meromrophic function on $\mathbf{C}$, and let $a_1, \ldots, a_q$ be q distinct points in $\mathbf{P}^1$. Then*

$$(q-2)T(f,r) \leq \sum_{j=1}^{q} \overline{N}(f, a_j, r) + o(T(f,r))$$

as $r \to \infty$ outside an exceptional set of radii of finite Lebesgue measure in $[0, \infty)$.

Here the use of $\overline{N}(f, a_j, r)$ rather than $N(f, a_j, r)$ indicates that we are counting the times f takes on the values a_j ignoring multiplicity, rather than counting multiplicity. This then implies the weaker inequality

$$(q-2)T(f,r) \leq \sum_{j=1}^{q} N(f, a_j, r) + o(T(f,r)),$$

which is lower bound on the number of times the function takes on the values a_j in terms of its characteristic function. Because we can ignore multiplicity and because of the appearance of the Euler characteristic of the Riemann sphere in the form of $-2T(f,r)$, we see that the Second Main Theorem is also the analog in transcendental function theory of the Riemann-Hurwitz formula for algebraic maps.

The Second Main Theorem is the highlight of Nevanlinna's theory and is significantly deeper than the First Main Theorem. Extending the First Main Theorem to relatively general geometric situations is usually not difficult, whereas analogs of the Second Main Theorem in complex algebraic geometry remain largely conjectures, although there are theorems in the important special cases of maps to Abelian varieties and projective spaces. Detailed treatments of Nevanlinna's theory for meromorphic functions on the complex plane can be found in [ChYe], [GoOs], [Hay], [JV], [Ne], [Ru]. Generalizations to higher dimensions can be found in [NO], [Sh].

I will conclude this section with a special case of a result of Valiron [Va].

Proposition 2.4 (Valiron). *Let $a_0, \ldots, a_d$ be meromorphic functions on $\mathbf{C}$ and let f be a meromorphic function on $\mathbf{C}$ such that*

$$\sum_{j=0}^{d} a_j f^j = 0.$$

Then, $T(f,r) \leq \sum_{j=0}^{d} T(a_j, r) + O(1)$.

Remark. Valiron proved Propostion 2.4 in the more general case that f might be multivalued. Multivalued functions which are alegebraic over the field of meromorphic functions on $\mathbf{C}$ are called **algebroid** functions. Propositon 2.4 in the form stated here was probably familiar before in the years before Valiron's paper and is implicit, for instance, in [Bl].

Proof. We may assume f and a_d are not identically zero. To estimate $m(f,\infty,r)$, observe that

$$\begin{aligned} d\log^+|f| = \log^+|f|^d &= \log^+\left(\left|\sum_{j=0}^{d-1}\frac{a_j}{a_d}f^j\right|\right) \\ &\leq \log^+\left|\frac{1}{a_d}\right| + \sum_{j=0}^{d-1}\left[\log^+|a_j| + \log^+|f|^j\right] + O(1) \\ &\leq \log^+|a_0| + \ldots \log^+|a_{d-1}| + \log^+\left|\frac{1}{a_d}\right| + (d-1)\log^+|f| + O(1). \end{aligned}$$

Hence $m(f,\infty,r) \leq m(a_0,\infty,r) + \ldots + m(a_{d-1},\infty,r) + m(a_d,0,r) + O(1)$.

To estimate $N(f,\infty,r)$, let b be the least common denominator of $a_0,\ldots,a_{d-1}$ so that b and ba_j are entire for $0 \leq j \leq d-1$. Then,

$$N(b,0,r) \leq \sum_{j=0}^{d-1} N(a_j,\infty,r).$$

On the other hand,

$$ba_d = -\sum_{j=0}^{d-1} ba_j\left(\frac{1}{f}\right)^{d-j}.$$

Therefore, any pole of f must be a zero of ba_d with at least the same multiplicity, and so $N(f,\infty,r) \leq N(ba_d,0,r)$. Hence,

$$N(f,\infty,r) \leq N(a_d,0,r) + N(b,0,r) \leq N(a_d,0,r) + \sum_{j=0}^{d-1} N(a_j,\infty,r).$$

The proof concludes by applying the First Main Theorem. □

2.2 A Differential Geometric Interpretation of the Nevanlinna Characterisic

Let $dz = dx + idy$ and $d\bar{z} = dx - idy$. If f is a function on $\mathbf{C}$, we define

$$\partial f = \frac{\partial f}{\partial z}dz \qquad \text{and} \qquad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

It is convenient to introduce the real derivatives

$$df = \partial f + \bar{\partial} f \qquad \text{and} \qquad d^c f = \frac{i}{4\pi}(\bar{\partial} f - \partial f).$$

The operator d^c was introduced by Griffiths. The i in the numerator makes the operator real, *i.e.,* invariant under complex conjugation if f is real valued. The 4π in the denominator is meant to minimize the appearance of powers of 2 and π in important geometric formulas, althought they cannot be eliminated entirely. Importantly,

$$dd^c f = \frac{i}{2\pi}\frac{\partial^2 f}{\partial z \partial \bar{z}}dz \wedge d\bar{z}.$$

Note that $\partial^2 f/\partial z\partial\bar{z}$ is, up to a constant factor (of 4), the Laplacian of f.

Exercise 2.5. *If $z = re^{i\theta}$, then $d\log|z|^2 = 2\frac{dr}{r}$ and $d^c\log|z|^2 = \frac{d\theta}{2\pi}$.*

Exercise 2.6. *Let S be the sphere in $\mathbf{R}^3$ centered at the origin with surface area 1. If the complex plane $\mathbf{C}$ is identified with S minus its north pole via stereographic projection and if $E \subset \mathbf{C}$ is a measurable subset, then the spherical area of E is given by*

$$\int_E \frac{r\,dr\,d\theta}{\pi(1+r^2)^2} = \int_E \frac{i}{2\pi}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}.$$

We define the **Fubini-Study** or **spherical area** form ω_{FS} form on $\mathbf{P}^1$ by

$$\omega_{\mathrm{FS}} = \frac{i}{2\pi}\frac{dz \wedge d\bar{z}}{\left(1+|z|^2\right)^2}.$$

Given a meromorphic function f, define its **spherical derivative** $f^{\#}$ by

$$f^{\#}(z) = \begin{cases} \dfrac{|f'(z)|}{1+|f(z)|^2} & \text{if } f(z) \neq \infty \\ \left|\left(\dfrac{1}{f}\right)'(z)\right| & \text{if } f(z) = \infty. \end{cases}$$

Exercise 2.7. *Let f be a meromorphic function on $\mathbf{C}$. Then,*

$$\int_{\mathbf{D}(t)} f^*\omega_{\mathrm{FS}} = \int_{\mathbf{D}(t)} \left(f^{\#}(x+iy)\right)^2 \frac{dx\,dy}{\pi}$$

measures the area of S coverby by $f(\mathbf{D}(t))$, counting multiplicity, where S is as in exercise 2.6.

Exercise 2.8. *For all $x > 0$, we have $\log^+ x \le \frac{1}{2}\log(1+x^2) \le \log^+ x + \frac{\log 2}{2}$, and hence*

$$m(f,\infty,r) \le \frac{1}{2}\int_0^{2\pi} \log\left(1+|f(re^{i\theta})|^2\right)\frac{d\theta}{2\pi} \le m(f,\infty,r) + \frac{\log 2}{2}.$$

Exercise 2.9. *Away from singularities $-dd^c \log f^{\#} = dd^c(1+|f|^2) = f^*\omega_{\mathrm{FS}}$.*

Theorem 2.10 (Green-Jensen Formula). *Let u be a function that is C^2 in a neighborhood of $\mathbf{D}(r)$ except at a discrete set of singularities Z, and assume that u is continuous at 0. Assume further that u satisfies the following three conditions:*

GJ 1. *u is absolutely integrable on $\partial\mathbf{D}(r)$.*

GJ 2. *$du \wedge \dfrac{d\theta}{2\pi}$ is absolutely integrable on $\mathbf{D}(r)$.*

GJ 3. *For every non-zero a in Z, $\displaystyle\lim_{\varepsilon\to 0}\int_{|z-a|=\varepsilon} u\,\frac{d\theta}{2\pi} = 0$.*

GJ 4. *$dd^c u$ is absolutely integrable on $\mathbf{D}(r)$.*

Then

$$\int_0^r \frac{dt}{t}\int_{\mathbf{D}(t)} dd^c u + \int_0^r \frac{dt}{t}\lim_{\varepsilon\to 0}\int_{S(Z,\varepsilon,t)} d^c u = \frac{1}{2}\int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi} - \frac{1}{2}u(0),$$

where $S(Z,\varepsilon,t)$ is the portion of $S(Z,\varepsilon)$ inside $\mathbf{D}(t)$. Here $S(z,\varepsilon)$ for ε small denotes the cycle consisting of circles of radius epsilon centered at the points in Z.

Proof. We will evaluate the integral

$$\int_{\mathbf{D}(r)} du \wedge \frac{d\theta}{2\pi}$$

two different ways: by using Stokes's Theorem and by using Fubini's Theorem. We remark that the integral is convergent by condition **GJ 4**.

If 0 is in Z, let $Z' = Z \setminus \{0\}$, and otherwise let $Z' = Z$. To apply Stokes's Theorem, note that

$$\int_{\mathbf{D}(r)} d\left(u\frac{d\theta}{2\pi}\right) = \lim_{\varepsilon \to 0} \int_{\mathbf{D}(r) - \left(\mathbf{D}(\varepsilon) \cup \mathbf{D}(Z',\varepsilon)\right)} d\left(u\frac{d\theta}{2\pi}\right),$$

where $D(Z', \varepsilon)$ denotes the formal sum of open discs of radius ε centered around the points of Z'. Because $d(d\theta) = 0$,

$$du \wedge \frac{d\theta}{2\pi} = d\left(u\frac{d\theta}{2\pi}\right)$$

and applying Stokes's Theorem, we get

$$\int_{\mathbf{D}(r)} d\left(u\frac{d\theta}{2\pi}\right) = \int_{\partial\mathbf{D}(r)} u\,\frac{d\theta}{2\pi} - \lim_{\varepsilon \to 0}\left[\int_{\partial\mathbf{D}(\varepsilon)} u\,\frac{d\theta}{2\pi} + \int_{S(Z',\varepsilon)} u\,\frac{d\theta}{2\pi}\right].$$

Now,

$$\lim_{\varepsilon \to 0} \int_{\partial\mathbf{D}(\varepsilon)} u\,\frac{d\theta}{2\pi} = u(0)$$

since u is assumed to be continuous at 0. The term

$$\lim_{\varepsilon \to 0} \int_{S(Z',\varepsilon)} u\,\frac{d\theta}{2\pi}$$

vanishes by assumption **GJ 3**.

Before applying Fubini, note that for any two C^2 functions α and β, we have, for degree reasons,

$$d\alpha \wedge d^c\beta = d\beta \wedge d^c\alpha.$$

So in particular, away from singularities,

$$du \wedge \frac{d\theta}{2\pi} = du \wedge d^c \log|z|^2 = d\log|z|^2 \wedge d^c u$$

by Exercise 2.5. The disc $\mathbf{D}(r)$ is fibered over the interval $(0, r)$ by the map $z \mapsto |z|$, and the form $d\log|z|^2$ is the pull back of the form $2dt/t$ on the interval $(0, r)$ under this mapping. Therefore, when we apply Fubini, we get

$$\int_{\mathbf{D}(r)} du \wedge \frac{d\theta}{2\pi} = 2\int_0^r \frac{dt}{t} \int_{\partial\mathbf{D}(t)} d^c u.$$

Thus,

$$\int_0^r \frac{dt}{t} \int_{\partial\mathbf{D}(t)} d^c u = \frac{1}{2} \int_{\partial\mathbf{D}(r)} u(re^{i\theta})\,\frac{d\theta}{2\pi} - \frac{1}{2}u(0).$$

Since for almost all t, the set of singularities Z does not intersect the boundary of $\mathbf{D}(t)$, we can therefore again apply Stokes's Theorem to the left hand side of the above equality to complete the proof. □

Theorem 2.11 (Ahlfors-Shimizu). $\displaystyle\int_0^r \frac{dt}{t}\int_{\mathbf{D}(t)} f^*\omega_{\mathrm{FS}} = T(f,r) + O(1).$

The proof is left as an exercise. Combine Exercise 2.9 with Thorem 2.10 and the definitions, and show that the singular term is the counting function of poles.

The above can be slightly jazzed up to handle maps to projective spaces. Let $f : \mathbf{C} \to \mathbf{P}^n$ be a holomorphic curve in $\mathbf{P}^n$ and represent f by an $n+1$ tuple $(f_0, \ldots, f_n)$ of entire functions without common zeros. Denote by

$$||f||^2 = |f_0|^2 + \ldots + |f_n|^2,$$

and beware that $||f||^2$ depends not only on f, but also on the choice of homogeneous coordinate functions $f_0, \ldots, f_n$. Then, the **Fubini-Study** derivative $f^\#$ of the map f is defined by

$$dd^c \log ||f||^2 = (f^\#)^2 \frac{i}{2\pi} dz \wedge d\bar{z}.$$

Note that if we change to another set of homogeneous cooredinates $g_j = hf_j$, where h is a non-vanishing entire function, then $||g||^2 = |h|^2||f||^2$, and thus

$$dd^c \log ||g||^2 = dd^c \log ||f||^2 + dd^c \log |h|^2 = dd^c \log ||f||^2$$

since $\log |h|^2$ is harmonic. Therefore, $dd^c \log ||f||^2$ and hence $f^\#$ does not depend on the choice of homogenous coordinate functions f_j, although it does depend on the choice of coordinates on $\mathbf{P}^n$. Similarly, if $Z = (Z_0, \ldots, Z_n)$ are homogeneous coordinates on $\mathbf{P}^n$, then the expression $dd^c \log ||Z||^2$ defines a positive $(1,1)$-form ω_{FS} on $\mathbf{P}^n$ known as the Fubini-Study form, and the associated Hermetian metric is called the Fubini-Study metric. A $(1,1)$-form ω is said to be **positive** if when it is written in local coordinates $(z^1, \ldots, z^n)$ in the form

$$\omega = \frac{i}{2\pi}\sum_{j=1}^{n}\sum_{k=1}^{n} \omega_{jk} dz^j \wedge d\bar{z}^k,$$

then the matrix (ω_{jk}) is positive definite. Thus, just as in the case of the Riemann sphere,

$$dd^c \log ||f||^2 = f^*\omega_{\mathrm{FS}}.$$

We therefore define the height or characteristic of a map to projective space by

$$T(f,r) = \int_0^r \frac{dt}{t}\int_{\mathbf{D}(t)} dd^c \log ||f||^2.$$

By Theorem 2.10, we can also write this as

$$T(f,r) = \frac{1}{2}\int_0^{2\pi} \log ||f(re^{i\theta})||^2 \frac{d\theta}{2\pi} - \frac{1}{2}\log ||f(0)||^2.$$

Without loss of generality, assume $f_0(0) \neq 0$. Then, by making use of the Jensen Formula (Theorem 2.1) applied to f_0, we can also write

$$T(f,r) = \frac{1}{2}\int_0^{2\pi} \log\left(1 + \sum_{j=1}^{\infty}\left|\frac{f_j}{f_0}(re^{i\theta})\right|^2\right) - \frac{1}{2}\log\left(1 + \sum_{j=1}^{n}\left|\frac{f_j}{f_0}(0)\right|^2\right) + N(f_0, 0, r).$$

Since

$$\log\left(1 + \sum_{j=1}^{\infty}\left|\frac{f_j}{f_0}\right|^2\right) \leq \sum_{j=1}^{n} \log^+\left|\frac{f_j}{f_0}\right|^2 + O(1),$$

we conclude

$$T(f,r) \le N(f_0,0,r) + \sum_{j=1}^{n} m\left(\frac{f_j}{f_0},\infty,r\right) + O(1).$$

Since the f_j are without common zeros,

$$N(f_0,0,r) \le \sum_{j=1}^{n} N\left(\frac{f_j}{f_0},\infty,r\right),$$

and hence

Theorem 2.12. *If $f : \mathbf{C} \to \mathbf{P}^n$ is a holomorphic curve represented by homogeneous coordinate functions $(f_0,\dots,f_n)$ where the f_j are entire without common zeros, then*

$$T(f,r) \le \sum_{j=1}^{n} T\left(\frac{f_j}{f_0},r\right) + O(1),$$

provided $f_0(0) \ne 0$.

Remark. The requirement that $f_0(0) \ne 0$ stated in Theorem 2.12 is only for convenience. It suffices that $f_0 \not\equiv 0$.

Finally, suppose $A \subset \mathbf{P}^N$ is an Abelian variety of dimension n embedded in the projective space $\mathbf{P}^N$ and $f : \mathbf{C} \to A$ is a holomorphic curve in A. Then, on the one had, we can consider f as a holomorphic curve in $\mathbf{P}^N$ and consider its characteristic function with respect to the Fubini-Study metric as defined above, which we can denote by

$$T_{\mathrm{FS}}(f,r) = \int_0^r \frac{dt}{t} \int_{\mathbf{D}(t)} f^*\omega_{\mathrm{FS}}.$$

On the other hand, representing A as a complex torus $\mathbf{C}^n/\Lambda$ and letting $(z^1,\dots,z^n)$ denote the coordinates on $\mathbf{C}^n$, then the $(1,1)$-form

$$\omega_{\mathrm{Flat}} = \frac{i}{2\pi} \sum_{j=1}^{n} dz^j \wedge d\bar{z}^j$$

is well-defined and positive on A. Thus, we could also consider the characteristic function defined by this natural metric,

$$T_{\mathrm{Flat}}(f,r) = \int_0^r \frac{dt}{t} \int_{\mathbf{D}(t)} f^*\omega_{\mathrm{Flat}}.$$

We also remark that if $\tilde{f} = (\tilde{f}_1,\dots,\tilde{f}_n)$ is a lift of f to $\mathbf{C}^n$, then

$$f^*\omega_{\mathrm{Flat}} = ||\tilde{f}'||^2 \frac{i}{2\pi} dz \wedge d\bar{z} \qquad \text{where } ||\tilde{f}'||^2 = \sum_{j=1}^{n} |\tilde{f}_j'|^2. \tag{1}$$

Because A is compact, the ratio $||\tilde{f}'||/f^{\#}$ is bounded above and below, and hence

$$T_{\mathrm{FS}}(f,r) = O\big(T_{\mathrm{Flat}}(f,r)\big) \qquad \text{and} \qquad T_{\mathrm{Flat}}(f,r) = O\big(T_{\mathrm{FS}}(f,r)\big). \tag{2}$$

2.3 Lemma on the Logarithmic Derivative

If f is a meromorphic function that is not identically zero, then f'/f is called the **logarithmic derivative** of f and is again a meromorphic function. Not every meromorphic function is a logarithmic derivative. For instance, logairthmic derivatives cannot have poles with multiplicity larger than one and they must have integer residues. Exploiting special properites of logarithmic derivatives is an important aspect of modern function theory. In this section, we will introduce a key contribution of Nevanlinna known as the Lemma on the Logarithmic Derivative.

To get a sense of the Logarithmic Derivative Lemma, let us first consider the case of a polynomial. If f is a polynomial, then f' has smaller degree than f, and so

$$\left|\frac{f'(z)}{f(z)}\right| \to 0 \text{ as } r \to \infty.$$

Even if f is a rational function $f = f_1/f_0$ with f_1 and f_0 polynomials, then $f'/f = f_1'/f_1 - f_0'/f_0$, and so

$$\left|\frac{f'(z)}{f(z)}\right| = \left|\frac{f_1'(z)}{f_1(z)} - \frac{f_0'(z)}{f_0'(z)}\right| \leq \left|\frac{f_1'(z)}{f_1(z)}\right| + \left|\frac{f_0'(z)}{f_0'(z)}\right| \to 0 \text{ as } r \to \infty.$$

For transcendental f, one cannot expect f'/f to go to zero as $z \to \infty$, as examples of the form $e^{P(z)}$ with $P(z)$ polynomial show. But, the phenomenon that f'/f is small relative to f persits for transcendental f in the following sense.

Theorem 2.13 (Nevanlinna's Lemma on the Logarithmic Derivative). *If $f \not\equiv 0$ is a meromorphic function on the complex plane $\mathbf{C}$, then*

$$m\left(\frac{f'}{f}, \infty, r\right) = o(T(f,r))$$

as $r \to \infty$ outside a set of exceptional radii of finite Lebesgue measure in $[0,\infty)$.

Nevanlinna's Second Main Theorem follows relatively straightforwardly from the Lemma on the Logarithmic Derivative. This deep fact about logarithmic derivatives is one of Nevanlinna's key contributions. I know of no short proof of the Logarithmic Derivative Lemma, and examples showing the necessity of the exceptional set of radii indicate that there likely is not an especially simple proof. Nevanlinna's proof is, however, elementary, consisting essentially of the Poisson-Jensen formula, a string of elementary estimates, and a calculus lemma. I will omit the proof of the Logarithmic Derivative Lemma here. Relatively simple proofs of the Logarithmic Derivative Lemma can be found in [Hay], [JV], [Ne], or [Lng 3]. Note that the right hand side of the inequality in the Logarithmic Derivative Lemma that results for the simplest proofs usually contains a $O(\log r)$ term, which is not $o(T(f,r))$ when f is a rational function. However, we already saw that Theorem 2.13 is true as stated for rational functions, and it is also not hard to see that if $\log r$ is not $o(T(f,r))$, then f is indeed a rational function. One can also give a proof that avoids the introduction of the $\log r$ term altogether; see *e.g.* [ChYe].

Corollary 2.14. *Let f be a meromorphic function on $\mathbf{C}$ such that its k-the derivative $f^{(k)} \not\equiv 0$. Then,*

$$T(f^{(k+1)}, r) = O(T(f,r))$$

as $r \to \infty$ outside an exceptional set of radii with finite Lebesgue measure in $[0,\infty)$.

Proof. We begin with the case $k = 0$. We have

$$N(f', \infty, r) \leq 2N(f, \infty, r)$$

since f' can only have poles at places where f has poles, and the multiplicity of the pole goes up by one in taking derivatives (and is hence at most double the multiplicity of the pole f has). On the other hand,

$$m(f',\infty,r) = m\left(f\frac{f'}{f},\infty,r\right) \leq m(f,\infty,r) + m\left(\frac{f'}{f},\infty,r\right) \leq T(f,r) + o(T(f,r))$$

as $r \to \infty$ outside an exceptional set. Here we have used the elementary fact that for two meromorphic functions g_1 and g_2,

$$m(g_1g_2,\infty,r) \leq m(g_1,\infty,r) + m(g_2,\infty,r).$$

Thus,

$$T(f,r) \leq 2T(f,r) + o(T(f,r)) + O(1) = O(T(f,r))$$

outside an exceptoinal set by Theorem 2.13. The inequality for higher k follows easily by induction. □

Remark. With the possible exception of equation (2), all the expressions of the form O or o in sections 2.1–2.3 can be worked out explicitly, see *e.g.* [ChYe].

2.4 Jet Bundles

Jet bundles are generalizations of tangent bundles. Kobayashi [Ko] attributes the introduction of the concept of jets and jet bundles to Ehresmann.

Let X be a complex analytic space and let x be a point of X. Let f be a holomorphic map from a small neighborhood of the origin in $\mathbf{C}$ such that $f(0) = x$. Choose local coordinates $z^1,\ldots,z^n$ around x in X. Then, $z^j \circ f$ are holomorphic functions and can be represented by convergent power series about the origin in $\mathbf{C}$. Given an integer $k \geq 0$, we define an equivalence relation $\sim_k$ on the set of such holomorphic maps by saying that $f \sim_k g$ if the first power series coefficients of order up to k of $z^j \circ f$ and $z^j \circ g$ agree for all j. When this happens we say f and g **osculate to order** k. The equivalence relation $\sim_k$ does not depend on the choice of local coordiantes $z^1,\ldots,z^n$. Note that since we required $f(0) = g(0) = x$, when $k = 0$, there is only one equivalence class. When $k = 1$, we have $f \sim_1 g$ if and only if f and g are tangent at x. An equivalence class under the relation $\sim_k$ is called a k**-jet** at x, and I denote by $J_{k,x}(X)$ the set of k-jets at x. When x is a smooth point of X, then clearly $J_{k,x}(X) \cong \mathbf{C}^{nk}$ where n is the dimension of X at x, since the jet is determined by the nk numbers consisting of the first through k-th power series coefficients for each of the n holomorphic functions $z^j \circ f$. Of course this isomorphism depends on the choice of local coordinates $z^1,\ldots,z^n$. The space $J_{1,x}(X)$ is nothing other than the tangent cone of X at x. The k**-th jet space** $J_k(X)$ is defined as a set as

$$J_k(X) = \bigcup_{x\in X} J_{k,x}(X).$$

The set $J_k(X)$ naturally carries the structure of a complex space with a natural projection map $p : J_k(X) \to X$. As mentioned above, over the non-singular part of X, the fibers are isomorphic to the complex vector space $\mathbf{C}^{nk}$, and above small enough open sets U in the non-singular part of X, we have

$$p^{-1}(U) \cong U \times \mathbf{C}^{nk}.$$

Such objects are called **holomorphic fiber bundles.** When $k > 1$, changing local coordintates results in non-linear transition functions on the fibers, and so the spaces $J_k(X)$ are not vector bundles.

If U is an open subset of $\mathbf{C}$ and $f : U \to X$ is a holomorphic map, then at each point z in U, the map f represents a jet in $J_{k,f(z)}(X)$; denote that jet by $j_kf(z)$. Thus, the holomorphic map

$f : U \to X$ induces a holomorphic map $j_k f : U \to J_k(X)$, and this map is called the k**-th jet** of f. When $k = 1$, the jet $j_1 f$ is the usual map induced by f to the tangent space of X, typically denoted df.

Jet spaces have played an important role in complex geometry from time to time and have been increasingly investigated in algebraic geometry and commutative algebra in recent times. See, for instance, the following references for more details on jets and jet spaces: [GrGr], [Gra], [De], [GoSm], [EiMu].

Jet bundles over complex tori are particularly simple because they are "trivial," meaning they are globally a product.

Proposition 2.15. *Let T be a (complete) complex torus of dimension n. Then, for each non-negative integer k,*

$$J_k(T) \cong T \times \mathbf{C}^{nk}.$$

Proof. Write $T = \mathbf{C}^n/\Lambda$ and choose coordinates on $\mathbf{C}^n$. Any holomorphic map $f : U \to T$ lifts to a holomorphic map $\tilde{f} : U \to \mathbf{C}^n$. The power series coefficients of the coordinate functions of $\tilde{f}$ uniquely determine the jets $j_k f$ and provide the isomorphism. □

Bloch's idea was to use this triviality and study the projections onto $\mathbf{C}^{nk}$ for k equal to the dimension of the Zariski closure of the image of f.

2.5 Bloch's Theorem

In lecture 1, we saw that if $f : \mathbf{C} \to X \subset T$ is a non-constant holomorphic curve in a subvariety X of a complete complex torus T, then X contains the translate of a non-trivial complex subtorus. However, we were unable to make a connection between the image of f and the translate of the subtorus, other than to say that the translate of the subtorus is contained in the closure of the image of f. Bloch's theorem allows us to refine that result for Abelian varieties and say that the closure of the image of f is itself the translate of an Abelian subvariety.

Theorem 2.16 (Bloch). *Let $f : \mathbf{C} \to A$ be a holomorphic curve in an Abelian variety. Then the Zariski closure of the image of f is the translate of an Abelian subvariety.*

Corollary 2.17. *Let $f : \mathbf{C} \to X \subset A$ be a holomorphic curve in a closed subvariety X of an Abelian variety A. Then, the Zariski closure of the image of f is the translate of an Abelian subvariety of A that is contained in X.*

I will give a formal proof of Bloch's theorem in the next lecture, but I will conclude this lecture with an outline of the structure of the proof. Let X be the Zariski closure in A of the image of f, let m be the dimension of X, and let n be the dimension of A. We want to show that X is the translate of an Abelian subvariety of A. The dimension of $J_m(X)$ (at least above the non-singular points of X) is $(m+1)m$. Since $n > m$ (or the theorem is trivial), $nm \geq \dim J_m(X)$. Bloch's crucial idea was to consider the projection $q_m : J_m(X) \to \mathbf{C}^{nm}$. We wil see that this is in fact a rational map. If dq_m fails to have maximal rank, then a certain generalized Wronskian vanishes, and this will allow us to see that X remains invariant under translation by a non-trivial Abelian subvariety of A. This allows us to quotient out by this Abelian subvariety and consider a smaller dimensional image. On the other hand, if dq_m has maximal rank, then q_m is a dominant rational map onto its image, which is a variety of the same dimension. Hence, any rational function on $J_m(X)$ is algebraic over the field of rational functions defining the map q_m. By embedding A in a large projective space $\mathbf{P}^N$, there exist finitely many rational functions ϕ_ℓ on X such that $T_{\mathrm{FS}}(f,r) = O\big(\max T(\phi_\ell \circ f, r)\big)$. Since the ϕ_ℓ are rational functions on X and hence on $J_m(X)$, they are algebraic over the field generated by the rational functions defining the map q_m. It will turn out that this will mean that the $\phi_\ell \circ f$ are algebraic over the field generated by $\tilde{f}_j^{(k)}$ for

$j = 1, \ldots, n$ and $k = 1, \ldots, m$, where the $\tilde{f}_j$ denote the coordinate functions of a lift $\tilde{f} : \mathbf{C} \to \mathbf{C}^n$. This means

$$T(\phi_\ell \circ f, r) = O\left(\max_{\substack{1 \le j \le n \\ 1 \le k \le m}} T(\tilde{f}_j^{(k)}, r) \right) = O(\max T(\tilde{f}_j, r)).$$

Thus we have $T_{\mathrm{FS}}(f, r) = T_{\mathrm{FS}}(p \circ \tilde{f}, r) = O(\max T(\tilde{f}_j, r))$, where $p : \mathbf{C}^n \to A$ is the universal covering map. As p is highly transcendental, we should expect $T_{\mathrm{FS}}(p \circ \tilde{f}, r)$ to be much larger than $T(\tilde{f}_j, r)$, and so we should expect to arrive at a contradiction, which we will do by explicitly comparing $T_{\mathrm{Flat}}(p \circ f, r)$ and $T(\tilde{f}_j, r)$.

3 The Proof of Bloch's Theorem, Related Recent Advances, Connections to Arithmetic and Algebraic Geometry, and Open Problems

3.1 The Proof of Bloch's Theorem

Proof of Theorem 2.16. Let $A = \mathbf{C}^n/\Lambda$ and let X be the Zarsiki closure in A of the image of $f : \mathbf{C} \to A$. Let m denote the dimension of X. Our goal is to show that X is the translate of an Abelian subvariety. Of course if $X = A$, there is nothing to show, so we henceforth assume that $m < n$.

The inclusion $\iota : X \hookrightarrow A$ induces inclusions $\iota_k : J_k(X) \hookrightarrow J_k(A)$ between the jet spaces. By Proposition 2.15, a choice of local coordinates on $\mathbf{C}^n$ induce projection mappings $q_k : J_k(A) \to \mathbf{C}^{nk}$. We consider the composition $q_k \circ \iota_k : J_k(X) \to \mathbf{C}^{nk}$, which by abuse of notation we continue to denote q_k. Fix a point x_0 in the non-singular locus X_{ns} of X, and without loss of generality assume $f(0) = x_0$. Choose the coordinates $z^1, \ldots, z^n$ on $\mathbf{C}^n$ and a small analytic open neighborhood U of x in A such that $z^1 = \ldots = z^n = 0$ is the point x_0, such that $z^1, \ldots, z^n$ are local coordinates in U, such that $z^1, \ldots, z^m$ are local coordinates in $U \cap X_{\mathrm{ns}}$, and such that $dz^1 \wedge \cdots \wedge dz^m$ does not vanish on $U \cap X_{\mathrm{ns}}$. The functions $z^{m+1}, \ldots, z^n$ can be viewed as functions on $U \cap X_{\mathrm{ns}}$, and thus there exist holomorphic functions $F^{m+1}(z^1, \ldots, z^m), \ldots, F^n(z^1, \ldots, z^m)$ such that on $U \cap X_{\mathrm{ns}}$,

$$z^j = F^j(z^1, \ldots, z^m) \quad \text{and} \quad dz^j = \sum_{i=1}^{m} \frac{\partial F^j}{\partial z^i} dz^i \quad \text{for } j = m+1, \ldots, n.$$

As X and A are algebraic varieties, and dz^j and dz^i are globally defined one forms on A and on X_{ns}, it follows that even though the F^j are only locally defined holomoprhic functions, that the $\partial F^j / \partial z^i$ are in fact rational functions on X.

We now examine in more detail the projection map in local coordinates

$$q_m : J_m(U \cap X_{\mathrm{ns}}) \to \mathbf{C}^{mn}.$$

The local coordinates $z^1, \ldots, z^m$ trivialize $J_m(U \cap X_{\mathrm{ns}})$ as

$$J_m(U \cap X_{\mathrm{ns}}) \cong (U \cap X_{\mathrm{ns}}) \times \mathbf{C}^{mn}.$$

We can therefore consider $q_m : J_m(U \cap X_{\mathrm{ns}}) \to \mathbf{C}^{nm}$ as a map from $(U \cap X_{\mathrm{ns}}) \times \mathbf{C}^{m^2} \to \mathbf{C}^{nm}$ since the same system of local coordinates $z^1, \ldots, z^n$ trivialize both $J_m(U)$ and $J_m(A)$. More precisely, the map q_m is written in local coordinates as

$$q_m \begin{pmatrix} z^1 & \xi^{1,(1)} & \cdots & \xi^{1,(m)} \\ \vdots & \vdots & \cdots & \vdots \\ z^m & \xi^{m,(1)} & \cdots & \xi^{m,(m)} \end{pmatrix} = \begin{pmatrix} \xi^{1,(1)} & \cdots & \xi^{1,(m)} \\ \vdots & \cdots & \vdots \\ \xi^{m,(1)} & \cdots & \xi^{m,(m)} \\ Q^{m+1,(1)} & \cdots & Q^{m+1,(m)} \\ \vdots & \cdots & \vdots \\ Q^{n,(1)} & \cdots & Q^{n,(m)} \end{pmatrix}. \tag{3}$$

To explain this notation, we have already defined $z^1, \ldots, z^m$ as local coordinates on $U \cap X_{\mathrm{ns}}$. The $\xi^{i,(j)}$ for $1 \leq i \leq m$ and $1 \leq j \leq m$ are the standard coordinates on $\mathbf{C}^{m^2}$, but we have used the double index $i, (j)$ and the parentheses around the second index to signify that the first index i indexes the local coordinates $z^1, \ldots, z^m$ and the second index (j) indicates the number of "derivatives" that correspond to that portion of the jet fiber. The first m rows of the map q_m is the identity mapping because $z^1, \ldots, z^n$ trivialize both $J_m(U)$ and $J_m(A)$. The functions $Q^{i,(j)}$ for $i = m+1, \ldots, n$ and $j = 1, \ldots, m$ are *a priori* locally defined holomorphic functions of the z^i and the $\xi^{i,(j)}$, which we shall now examine more closely.

Proposition 3.1. *The $Q^{i,(j)}$ are polynomials in $\xi^{k,(\ell)}$ whose coefficients are partial derivatives of the F^i of order at most j, which themselves are rational functions on X.*

Proof. We have already seen that the first partial derivatives of the F^i are rational functions on X. It follows easily that higher partials must also be rational functions on X. For instance, since $\partial F^i / \partial z^j$ is a rational function on X, then

$$d \frac{\partial F^i}{\partial z^j} = \sum_{k=1}^{m} \frac{\partial^2 F^i}{\partial z^k \partial z^j} dz^k$$

is a rational 1-form on X. The dz^k are also rational one forms on X. Thus, the partial derivatives $\partial^2 F^i / \partial z^k \partial z^j$ must also be rational functions by Cramer's rule.

That the $Q^{i,(j)}$ are polynomials is just the chain rule, but the notation gets cumbersome for large j. I illustrate the first few cases. Recall that for $i = m+1, \ldots, n$,

$$z^i = F^i(z^1, \ldots, z^m).$$

If we differentiate this equation, we get

$$\frac{dz^i}{dt} = \sum_{k=1}^{m} \frac{\partial F^i}{\partial z^k} \frac{dz^k}{dt}.$$

The term dz^k/dt is represented by the local coordinate $\xi^{k,(1)}$ on the fiber of $J_m(U)$. Thus,

$$Q^{i,(1)} = \sum_{k=1}^{m} \frac{\partial F^i}{\partial z^k} \xi^{k,(1)}.$$

For $j = 2$, we need to compute

$$\frac{d^2 z^i}{dt^2} = \sum_{k=1}^{m} \frac{\partial F^i}{\partial z^k} \frac{d^2 z^k}{dt^2} + \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} \frac{\partial^2 F^i}{\partial z^\alpha z^\beta} \frac{dz^\alpha}{dt} \frac{dz^\beta}{dt}.$$

This exactly means

$$Q^{i,(2)} = \sum_{k=1}^{m} \frac{\partial F^i}{\partial z^k} \xi^{k,(2)} + \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} \frac{\partial^2 F^i}{\partial z^\alpha z^\beta} \xi^{\alpha,(1)} \xi^{\beta,(1)}. \qquad \square$$

Continuation of the proof of Theorem 2.16. Observe that $q_m : J_m(X_{\mathrm{ns}}) \to \mathbf{C}^{nm}$ is a map from an $(m+1)m$ dimensional space to an nm dimensional space with $n \geq m+1$. We now want to study the derivative dq_m. To ease notation, for $i = 1, \ldots, m$ and $j = m+1, \ldots, n$, we let

$$P_i^j = \frac{\partial F^j}{\partial z^i},$$

and recall that the P_i^j are rational functions on X.

Proposition 3.2. *If there exists $\delta > 0$ such that $(dq_m)_{j_m(z)}$ never has maximal rank for all $|z| < \delta$, then there exist complex constants $c_1, \ldots, c_m$ such that*

$$\sum_{i=1}^{m} c_i (P_i^j \circ f)' \equiv 0 \text{ for } j = m+1, \ldots, n.$$

Proof. By (3),

$$dq_m = \left(\begin{array}{ccc|c} & 0 & & I \\ \hline \frac{\partial Q^{m+1,(1)}}{\partial z^1} & \cdots & \frac{\partial Q^{m+1,(1)}}{\partial z^m} & \\ \vdots & \cdots & \vdots & \\ \frac{\partial Q^{m+1,(m)}}{\partial z^1} & \cdots & \frac{\partial Q^{m+1,(m)}}{\partial z^m} & \\ \vdots & \cdots & \vdots & * \\ \frac{\partial Q^{n},(1)}{\partial z^1} & \cdots & \frac{\partial Q^{n,(1)}}{\partial z^m} & \\ \vdots & \cdots & \vdots & \\ \frac{\partial Q^{n,(m)}}{\partial z^1} & \cdots & \frac{\partial Q^{n,(m)}}{\partial z^m} & \end{array}\right)$$

Thus, dq_m fails to have maximial rank if and only if the matrix

$$\begin{pmatrix} \frac{\partial Q^{m+1,(1)}}{\partial z^1} & \cdots & \frac{\partial Q^{m+1,(1)}}{\partial z^m} \\ \vdots & \cdots & \vdots \\ \frac{\partial Q^{m+1,(m)}}{\partial z^1} & \cdots & \frac{\partial Q^{m+1,(m)}}{\partial z^m} \\ \vdots & \cdots & \vdots \\ \frac{\partial Q^{n},(1)}{\partial z^1} & \cdots & \frac{\partial Q^{n,(1)}}{\partial z^m} \\ \vdots & \cdots & \vdots \\ \frac{\partial Q^{n,(m)}}{\partial z^1} & \cdots & \frac{\partial Q^{n,(m)}}{\partial z^m} \end{pmatrix}$$

has rank $< m$.

We now compute $\partial Q^{j,(k)}/\partial z^i$ and evaluate at $j_m f$. We write out what happens for $k = 1$ and 2. For $k = 1$, we have

$$Q^{j,(1)} = \sum_{\alpha=1}^{m} \frac{\partial F^j}{\partial z^\alpha} \xi^{\alpha,(1)},$$

and so by equality of mixed partials,

$$\frac{\partial Q^{j,(1)}}{\partial z^i} = \sum_{\alpha=1}^{m} \frac{\partial F^j}{\partial z^i \partial z^\alpha} \xi^{\alpha,(1)} = \sum_{\alpha=1}^{m} \frac{\partial P_i^j}{\partial z^\alpha} \xi^{\alpha,(1)}.$$

We thus see

$$\frac{\partial Q^{j,(1)}}{\partial z^i} \circ j_m f = \sum_{\alpha=1}^{m} \frac{\partial P_i}{\partial z^\alpha} \circ f (z^\alpha \circ f)' = (P_i^j \circ f)'.$$

For $k = 2$,

$$\begin{aligned}\frac{\partial Q^{j,(2)}}{\partial z^i} &= \sum_{\alpha=1}^{m} \frac{\partial^2 F^j}{\partial z^i \partial z^\alpha} \xi^{\alpha,(2)} + \sum_{\alpha=1}^{m}\sum_{\beta=1}^{m} \frac{\partial^3 F^{m+1}}{\partial z^i \partial z^\alpha \partial z^\beta} \xi^{\alpha,(1)} \xi^{\beta,(1)} \\ &= \sum_{\alpha=1}^{m} \frac{\partial P_i}{\partial z^\alpha} \xi^{\alpha,(2)} + \sum_{\alpha=1}^{m}\sum_{\beta=1}^{m} \frac{\partial^2 P_i}{\partial z^\alpha \partial z^\beta} \xi^{\alpha,(1)} \xi^{\beta,(1)},\end{aligned}$$

and thus, $\partial Q^{j,(2)}/\partial z^i \circ j_m f = (P_i \circ f)''$. Continuing in this manner, we find that $(dq_m)_{j_m f}$ fails to have maximal rank for all z in a neighborhood of the origin if and only if the generalized Wronskian matrix

$$\begin{pmatrix} (P_1^{m+1} \circ f)' & \cdots & (P_m^{m+1} \circ f)' \\ \vdots & \cdots & \vdots \\ (P_1^{m+1} \circ f)^{(m)} & \cdots & (P_m^{m+1} \circ f)^{(m)} \\ \vdots & \cdots & \vdots \\ (P_1^{n} \circ f)' & \cdots & (P_m^{n} \circ f)' \\ \vdots & \cdots & \vdots \\ (P_1^{n} \circ f)^{(m)} & \cdots & (P_m^{n} \circ f)^{(m)} \end{pmatrix}$$

fails to have maximal rank for z in a neighborhood of 0.

The proof of the proposition is completed by the following lemma on generalized Wronskians.

Lemma 3.3. *Let $(g_1^1 \dots, g_1^m), \dots, (g_L^1, \dots, g_L^m)$ be L m-tuples of meromorphic functions on a domain D in $\mathbf{C}$. If there do not exist complex constants $c_1, \dots, c_m$ not all zero such that*

$$c_1 g_\ell^1 + \cdots + c_m g_\ell^m \equiv 0 \quad \textit{for all } \ell = 1, \dots, L,$$

then the generalized Wronskian matrix

$$\begin{pmatrix} g_1^1 & \cdots & g_1^m \\ (g_1^1)' & \cdots & (g_1^m)' \\ \vdots & \cdots & \vdots \\ (g_1^1)^{(m-1)} & \cdots & (g_1^m)^{(m-1)} \\ \vdots & \cdots & \vdots \\ g_L^1 & \cdots & g_L^m \\ (g_L^1)' & \cdots & (g_L^m)' \\ \vdots & \cdots & \vdots \\ (g_L^1)^{(m-1)} & \cdots & (g_L^m)^{(m-1)} \end{pmatrix}$$

fails to have maximal rank at at most a discrete subset of D.

Proof. We proceed by induction on m. If $m = 1$, then the lemma is obvious. Now if $m > 1$, then whether or not the matrix has maximal rank is determined by the simultaneous vanishing of a finite number of sub-determinants, which are meromorphic functions on D. Thus, if the matrix fails to have maximal rank at more than a discrete subset of U, it fails to have maximal rank everywhere in U. In this case, the columns of the matrix must be linearly dependent over the field of meromorphic functions on U. Thus, we can find meromorphic functions $c_1(z), \ldots, c_m(z)$, not all identically zero, such that

$$\sum_{i=1}^{m} c_i(z)(g_\ell^i)^{(k)}(z) \quad \text{for all } \ell = 1, \ldots, L \text{ and all } k = 0, \ldots, m-1. \tag{4}$$

Without loss of generality, assume $c_m(z) \equiv 1$. Then, differentiating (4), we get

$$0 \equiv \sum_{i=1}^{m} c_i(g_\ell^i)^{(k+1)} + \sum_{i=1}^{m-1} c_i'(g_\ell^i)^{(k)} = \sum_{i=1}^{m-1} c_i'(g_\ell^i)^{(k)} \quad \text{for } \ell = 1, \ldots, L \text{ and } k = 0, \ldots, m-2.$$

Either $c_i' \equiv 0$ for all i, in which case we are done, or the lemma follows by induction since the smaller generalized Wronskian

$$\begin{pmatrix} g_1^1 & \cdots & g_1^{m-1} \\ (g_1^1)' & \cdots & (g_1^{m-1})' \\ \vdots & \cdots & \vdots \\ (g_1^1)^{(m-2)} & \cdots & (g_1^{m-1})^{(m-2)} \\ \vdots & \cdots & \vdots \\ g_L^1 & \cdots & g_L^{m-1} \\ (g_L^1)' & \cdots & (g_L^{m-1})' \\ \vdots & \cdots & \vdots \\ (g_L^1)^{(m-2)} & \cdots & (g_L^{m-1})^{(m-2)} \end{pmatrix}$$

fails to have maximal rank everywhere in U.

This completes the proof of Lemma 3.3 and Proposition 3.2. □

Now one of two things happens.

Lemma 3.4. *If there exists a $\delta > 0$ such that $(dq_m)_{j_m f(z)}$ does not have maximal rank for any $|z| < \delta$, then the dimension of the algebraic group $G = \{a \in A : X + a = X\}$ is positive dimensional, and in particular X is invariant by the identity component G^0 of G, which is a non-trivial Abelian subvariety of A.*

If we are in the case of Lemma 3.4, then we may quotient out by G_0 and consider the induced holomorphic curve

$$f \to X/G^0 \subset A/G^0.$$

If $X/G^0 = A/G^0$, then X is the translate of an Abelian subvariety in A and the proof of Theorem 2.16 is complete. Otherwise, we may repeat our analysis above on $X/G^0 \subset A/G^0$ and continue until we either end up showing X is the translate of an Abelian subvariety or that the Wronskian in Lemma 3.4 does not vanish identically.

Proof of Lemma 3.4. By Proposition 3.2, there exist constants $c_1, \ldots, c_m$ not all zero, such that for $j = m+1, \ldots, n$, we have

$$c_1(P_1^j \circ f)' + \ldots + (c_m P_m^j \circ f)' \equiv 0.$$

Integrating these relations, we find constants $c_{m+1}, \ldots, c_n$ such that for $j = m+1, \ldots, n$, we have

$$c_1 P_1^j \circ f + \ldots + c_m P_m^j \circ f = c_j.$$

However, as the P_i^j are rational functions on X and the image of f is dense in X, we conclude that

$$c_1 P_1^j + \ldots + c_m P_m^j = c_j$$

on X.

Let ν be the unique vector field on A such that

$$\nu(x_0) = \sum_{i=1}^{n} c_i \frac{\partial}{\partial z^i}\bigg|_{x_0}.$$

The vector field ν is obtained by

$$\nu(x) = (d\tau_{x-x_0})_{x_0} \left(\sum_{i=1}^{m} c_i \frac{\partial}{\partial z^i}\bigg|_{x_0} \right),$$

where $\tau_{x-x_0} : A \to A$ denotes translation by $x - x_0$. I claim ν is tangent to X. Indeed, consider what happens in $U \cap X$. Here,

$$\nu z^j = \sum_{i=1}^{n} c_i \frac{\partial z^j}{\partial z^i} = c_j$$

since the z^i are local coordinates in U. On the other hand, if $j > m$, then

$$\nu F^j = \sum_{i=1}^{n} c_i \frac{\partial F^j}{\partial z^i} = \sum_{i=1}^{m} c_i P_i^j = c_j$$

since the F^j do not depend on $z^{m+1}, \ldots, z^n$. Hence, on $X \cap U$, we have

$$\nu z^j = \nu F^j \quad \text{for } j = m+1, \ldots, n.$$

Since $X \cap U$ is defined by $z^j = F^j(z_1, \ldots, z_m)$ in $X_{\text{ns}} \cap U$, this exactly means that ν is tangent to X there. Since the set of x in X where ν is tangent to X is an algebraic subvariety of X, we must have ν tangent to X everywhere in X. This exactly means that X is left invariant by the one parameter subgroup of A associated to ν. □

Note that so far, we have not used the fact that the holomorphic curve f is defined on all of $\mathbf{C}$.

Continuation of the proof of Theorem 2.16. As explained above, if X is not the translate of an Abelian subvariety, we may assume that dq_m has maximal rank at $j_m f(z_0)$ for some small z_0. But this means that the rational functions $Q^{m+1,(1)}, \ldots, Q^{m+1,(m)}$ generically form local coordinates for X. In other words, the map $q_m : J_m(X) \to \mathbf{C}^{nm}$ is algebraic and dq_m generically has maximal rank. Therefore, the image of q_m has the same dimension as $J_m(X)$. Hence the field of rational functions on $J_m(X)$ is a finite algebraic extension of the field of coordinate functions of q_m; see, *e.g.* [NO, Th. 6.2.9].

If ϕ is a rational function on X, and hence also on $J_m(X)$, it is algebraic over the field

$$\mathbf{C}(\xi^{1,(1)}, \ldots, \xi^{m,(1)}, \ldots, \xi^{1,(m)}, \ldots, \xi^{m,(m)}, Q^{m+1,(1)}, \ldots, Q^{m+1,(m)}, \ldots, Q^{n,(1)}, \ldots, Q^{n,(m)}).$$

Composing with $j_m f$, this tells us that $\phi \circ f$ is algebraic over the field

$$\mathbf{C}((z^1 \circ f)', \ldots, (z^n \circ f)', \ldots, (z^1 \circ f)^{(m)}, \ldots, (z^n \circ f)^{(m)}).$$

Since X is projective, we can embed X in projective space $\mathbf{P}^N$, and let $T_{\text{FS}}(f, r)$ be the characteristic function of f obtained via the pull-back of the Fubini-Study form on $\mathbf{P}^N$. By Theorem 2.12, we can therefore find rational functions $\phi_1, \ldots, \phi_N$ on X such that $T_{\text{FS}}(f, r) = O\big(\max T(\phi_\ell \circ f, r)\big)$. By Theorem 2.4, for each function ϕ_ℓ, we have

$$T(\phi_\ell \circ f, r) = O\Big(\max T\big((z^j \circ f)^{(k)}, r\big)\Big).$$

Hence,

$$T_{\mathrm{FS}}(f,r) = O\big(\max T(z^j \circ f, r)\big)$$

as $r \to \infty$ outside an exceptional set by Corollary 2.14. By equation (2), we thus conclude

$$T_{\mathrm{Flat}}(f,r) = O\big(\max T(z^j \circ f, r)\big)$$

as $r \to \infty$ outside an exceptional set. But, using Theorem 2.10 and the definition of T_{Flat}, we have

$$\frac{1}{2}\sum_{j=1}^{n}\int_0^{2\pi} |z^j \circ f(re^{i\theta})|^2 \,\frac{d\theta}{2\pi} = \int_0^r \int_{\mathbf{D}(t)} dd^c \sum_{j=1}^{n} |z^j \circ f|^2 + O(1) = T_{\mathrm{Flat}}(f,r) + O(1).$$

Thus,

$$\begin{aligned}
\sum_{j=1}^{n} T(e^{z^j \circ f}, r) &= \sum_{j=1}^{n}\int_0^{2\pi} \log^+ |e^{z^j \circ f(re^{i\theta})}| \,\frac{d\theta}{2\pi} \\
&\le \sum_{j=1}^{n}\int_0^{2\pi} \log^+ e^{|z^j \circ f(re^{i\theta})|} \,\frac{d\theta}{2\pi} \\
&= \sum_{j=1}^{n}\int_0^{2\pi} |z^j \circ f(re^{i\theta})| \,\frac{d\theta}{2\pi} \\
&\le \sum_{j=1}^{n}\left[\int_0^{2\pi} |z^j \circ f(re^{i\theta})|^2 \,\frac{d\theta}{2\pi}\right]^{1/2} = O(T_{\mathrm{Flat}}(f,r)).
\end{aligned}$$

Hence,

$$\sum_{j=1}^{n} T(e^{z^j \circ f}, r) = O\big(\max T(z^j \circ f, r)\big),$$

as $r \to \infty$ outside an exceptional set. However, $z^j \circ f$ is the logarithmic derivative of $e^{z^j \circ f}$, and so Theorem 2.13 says

$$T(z^j \circ f, r) = m(z^j \circ f, \infty, r) = o\big(T(e^{z^j \circ f}, r)\big)$$

as $r \to \infty$ outside an exceptional set. We therefore conclude

$$\sum_{j=1}^{n} T(e^{z^j \circ f}, r) = o\big(\max T(e^{z^j \circ f}, r)\big),$$

as $r \to \infty$ outside an exceptional set, which is finally a contradiction. □

Historical Commentary. My exposition of the proof of Theorem 2.16 is based on the presentations in [NO], [Ko], and [Ru]. The overall structure of the proof of Theorem 2.16 given here is basically the same as in Bloch's original paper [Bl]. However, Bloch only treats the case that $\dim X = 2$ in some level of detail, and many mathematicians feel that Bloch's proof lack's sufficient detail and rigor. As here, Bloch projects onto the fiber of the m-th jet space and argues that a Wronskian being non-zero results in a contradiction, arrived at in exactly the same way as above. Where Bloch is sketchy is in the argument that the vanishing of his Wronskian implies that X is the translate of an Abelian subvariety. Bloch seems to regard as obvious the fact that if the Wronskian vanishes and X is not the translate of an Abelian subvariety, then the problem can be reduced to a smaller dimensional case where the corresponding Wronskian does not vanish. Ochiai [O] filled out most of Bloch's argument and brought it up to modern standards of rigor, though he also did not feel he had a complete proof of the step where we made use of Lemma 3.4 when $\dim X > 2$. Noguchi [No 1, pp. 227] credits Mark Green with giving a convincing proof of this step (Lemma 6.3.10 in [NO]) in a talk at the 1978 Taniguchi Symposium. Because of this history, many mathematicians refer to what I have called Bloch's Theorem as Bloch's Conjecture or the Bloch-Ochiai Theorem. There is no doubt that Green and Ochiai made decisive and important contributions by filling in the (perhaps substantial) gaps in Bloch's exposition, but following Siu [Si 1], [Si 2], I have chosen to refer to this as Bloch's theorem since the argument outlined by Bloch was eventually made to work rigorously and because Bloch is responsible for the key idea of showing that if the theorem is false, one eventually lands in a situation where the projection onto the jet fiber is a non-degenerate rational map. Similarly, the fact that every simply connected planar domain, other than the plane itself, maps conformally onto the unit disc is commonly referred to as the "Riemann Mapping Theorem," despite the fact that Riemann's proof contained a significant gap that was only filled much later.

3.2 Related Recent Advances

A group variet G is called a semi-Abelian variety if it fits into an exact sequence of algebraic morphisms

$$1 \to \mathbf{G}_m^k \to G \to A \to 1,$$

where A is an Abelian variety.

Theorem 3.5 (Noguchi). *The Zariski closure of the image of a holomorphic curve in a semi-Abelian variety is the translate of a semi-Abelian subvariety.*

Theorem 3.5 is due to Noguchi [No 1]. Most of the proof we have given for Bloch's conjecture works in this more general setting. However, in equation 2, we used the compactness of the Zariski closure of f in a fundamental way. Thus, something else needs to be done at this point. See the work of Noguchi and Winkelmann [NW] for the non-algebraic quasi-tori. An important part of the proof in the semi-Abelian and quasi-torus cases is studying the natural compactification one obtains by completing $\mathbf{G}_m$ to $\mathbf{P}^1$.

One can also ask what happens if a holomorphic curve omits a set in an Abelian variety.

Theorem 3.6 (Siu Yeung). *Let $f : \mathbf{C} \to A$ be a non-constant holomorphic curve in an Abelian variety A and let D be an ample divisor in A. Then, the image of f intersects D.*

Theorem 3.6 was a conjecture of Lang that was open for quite a while. It was proven by Siu and Yeung [SY 1] using an argument very similar to the proof of Bloch's theorem. The additional ingredient is something called a **logarithmic jet differential.** I first describe a jet differential. A k-jet differential is a globally defined object that in local coordinates $(z^1, \ldots, z^n)$ can be written as a linear combination of expressions of the form

$$(dz^1)^{m_{1,1}} \cdots (dz^n)^{m_{n,1}} \cdots (d^k z^1)^{m_{1,k}} \cdots (d^k z^m)^{m_{n,k}}.$$

If f is a holomorphic curve, then f naturally pulls back jet differentials. So for example, the above expression pulls back to

$$([f^1]')^{m_{1,1}} \cdots ([f^n]')^{m_{n,1}} \cdots ([f^1]^{(k)})^{m_{1,k}} \cdots ([f^n]^{(k)})^{m_{n,k}},$$

where $f^1, \ldots, f^n$ are local coordinate functionss representing f. If we allow poles, then we get what is know as either a rational or meromorphic jet differential, depending on whether we are working on complex geometry or algebraic geometry. If the poles are only of a special type, namely like generalized logarithmic derivatives, so we allow things of the form

$$\left(\frac{d^k z^j}{z^j}\right)^m$$

as factors, then we say the rational or meromorphic jet differential is a logarithmic jet differential. In studying holomorphic curves, logarithmic jet differentials are useful because the logarithmic derivative lemma can be applied to pull-backs of logarithmic jet differentials by holomorphic curves. Siu and Yeung use the fact that if D is an ample divisor on an Abelian variety A, then D can be defined by the vanishing of a theta function on the universal covering $\mathbf{C}^n$ of A. They use the theta function to construct a logarithmic jet differential and because the growth rate of the theta function is not too large, if a non-constant holomorphic curve omits an ample divisor, they can reach a contradiction in a similar way to the end of our proof of Bloch's theorem.

Finally, an analog of Nevanlinna's Second Main Theorem has been proven for divisors in semi-Abelian varieties that gives a quantitative version of Theorem 3.6, namely a measure of how often a non-constant holomorphic curve must encounter an ample divisor in terms of the growth of its characteristic function. I will not make a precise statment here; see [SY 2] (and also its erratum [SY 3]), [McQ], [NWY 1] and [NWY 1].

3.3 Connections to Arithmetic Algebraic Geometry

Let X be a non-singular projective algebraic variety defined over the complex numbers and let D be a, possibly empty, effective divisor on X with at worst normal crossing singularities.

Define the **analytic special set** Z_{anal} to be the smallest subvariety of X such that the image of every non-constant holomorphic curve $f : \mathbf{C} \to X \setminus D$ is contained in Z_{anal}. Bloch's Theorem says that if X is a subvariety of an Abelian variety and $D = 0$, then Z_{anal} is the smallest subvariety of X that contains all translates of non-trivial Abelian subvarities contained in X. It turns out that this is a finite union of translates of Abelian subvarieties. The **geometric special set** Z_{geom} is the smallest subvariety of X such that every non-constant rational image of a group variety in $X \setminus D$ is contained in Z_{geom}. If X and D are defined over a field k which is finitely generated over the rational numbers $\mathbf{Q}$, then the **arithmetic special set** is defined to be the smallest subvariety Z_{arith} of X such that for every finitely generated field extension F over k, at most finitely many F-rational points of X which are integral with respect to D are contained outside Z_{arith}. If $D = 0$, then integral with respect to D simply means F-rational.

Recall that a non-singular projective variety X is said to be **pseudo-canonical** (or **of general type**) if some multiple of the canonical bundle K_X is **pseudo-ample** (or **big**), which means that the global sections of some tensor multiple of K_X gives rise to a projective embedding of a Zariski open subset of X into projective space. If D is a divsior on X with at worst normal crossing singularities, then the pair (X, D) is called **log-pseudo-canonical** (or **of log general type**) if the meromorphic sections of some multiple of K_X with logarithmic poles along D give rise to an embedding of a Zariski open subset of $X \setminus D$ into some projective space.

Conjecture 3.7 (Strong Lang Conjecture). *Let X be a non-singular projective algebraic variety and let D be a, possibly empty, effective divisor on X with at worth normal crossing singularities. Assume that both X and D are defined over a field k which is finitely generated over $\mathbf{Q}$. Then $Z_{\text{anal}} = Z_{\text{geom}} = Z_{\text{arith}}$. Moreover, the special set can also be defined as the Zariski closure of all positive dimension proper subvarieties of X which are either not pseudo-canonical or not log-pseudo cananical, depending on whether D is zero or not. Moreover, if X is pseudo-canonical or (X, D) is log-pseudo-canonical, then the special set is a proper subvariety.*

The strong Lang conjecture is true for subvarieties of semi-Abelian varieties. I will explain this only for Abelian varieties and when $D = 0$. The Kawamata Structure Theorem [Ka] tells us that if X is a subvariety of an Abelian variety that is not itself a translate of an Abelian subvariety, then X is pseudo-canonical, and moreover the geometric special set is a finite union of translates of Abelian subvarities contained in X. Bloch's Theorem tells us that this same special subvariety is the analytic special set, and Falting's Theorem [Fa] tells us that it is the arithmetic special set. Note that the arithmetic results have also been extended to subvarieties of semi-Abelian varieties: [Vo]. Our proof of Bloch's Theorem made fundamental use of derivatives, which is a tool unavailable in arithmetic geometry. Thus the proofs of the arithmetic results are very different from what we have done here. However, one can give a proof of Bloch's Theorem that is very similar in structure to the arithmetic proofs; this was done by McQuillan [McQ].

Precious little is known about pseudo-canonical varities that are not closely related to subvarieties of Abelian varities. Many mathematicians are skeptical of the strong Lang conjecture. McQuillan has recently come up with a mixed characteristic example that he believes is evidence against the strong Lang conjecture. Lang also formulated a weaker version of his conjecture:

Conjecture 3.8 (Weak Lang Conjecture). *If (X, D) is as in the strong Lang conjecture and log-pseduo-canonical, then*

(i) ***Log Green-Griffiths Conjecture*** *[GrGr]: Every holomorpic curve contained in $X \setminus D$ is contained in a proper subvariety of X (which may depend on f;) and*
(ii) for every field F wich is finitely generated over k, the F-rational points of X which are integral with respect to D are contained in a proper algebraic subvariety of X (which may depend on F).

Both the weaker and stronger versions of Lang conjecture have striking consequences in terms of uniformly bounding the number of rational points on curves of genus at least two; see [CHM].

3.4 Open Problems

Conjecture 3.9. *Let X be a subvariety of an Abelian variety A. Let d_X denote the Kobayashi pseudo-distance on X. Let $x_1 \neq x_2$ be points in X. If $d_X(x_1, x_2) = 0$, then x_1 and x_2 are contained in the translate of an Abelian subvariety of A contained in X.*

Maybe more coming soon...

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