HYPERBOLIC p-ADIC ANALYTIC SPACES

A Dissertation
Presented to the Faculty of the Graduate School
of
Yale University
in Candidacy for the Degree of
Doctor of Philosophy

by
William Allen Cherry

May 1993

INTRODUCTION

Recall that a complex analytic space X is called **Brody hyperbolic** if the only analytic maps $f: \mathbf{C} \to X$ are the constant maps. Serge Lang has conjectured that this property is related to the Diophantine property of a variety being Mordellic and to the algebro-geometric property of a variety being of general type. Let F_0 be a field which is finitely generated over the rational numbers $\mathbf{Q}$, and let X be a projective algebraic variety defined over F_0. Then, X is said to be of **general type**, or **pseudo canonical**, if the canonical divisor class on X is pseudo ample, which means that some multiple of it gives rise to a projective embedding of a Zariski open subset of X. Lang calls X **Mordellic** if X has only finitely many rational points over every field F which is finitely generated over F_0. Then, one can state Lang's conjecture as follows: (see [**L2**] Conjecture VIII.1.2)

Conjecture (Lang). *The following conditions are equivalent for a projective variety X, defined over a subfield of the complex numbers which is finitely generated over the rationals.*

(a) Considered as a complex analytic space, X is Brody hyperbolic.
(b) X is Mordellic.
(c) Every subvariety of X is of general type (pseudo canonical).

The philosophy behind this conjecture is that if a variety contains a subvariety which is the rational image of a group variety, then the variety will contain the image of a non-constant holomorphic map because the (Lie group) exponential maps give non-constant holomorphic maps into group varieties. Similarly, group varieties have infinitely many rational points once they have a rational point of infinite order because the group operation is algebraic. Therefore, the above conjecture philosophically says that if any of the above three properties are true, then the variety contains no rational images of group varieties, and this should then imply the other two properties. Lang's conjecture is true for smooth projective curves: curves of genus ≥ 2 are Brody hyperbolic, which is an easy consequence of the Uniformization Theorem and Liouville's Theorem; such curves are Mordellic by Faltings's Theorem (the Mordell Conjecture); and curves of genus ≥ 2 are clearly of general type since their canonical class is ample.

Because p-adic analysis is more algebraic than complex analysis, one often looks at this branch of mathematics when trying to understand the relationship between an analytic and an algebraic property. Thus in the effort to better understand the connection between the analytic property of hyperbolicity and the Diophantine property of Mordellicity, Lang suggested searching for a p-adic analytic property analogous to Brody hyperbolicity.

The most obvious analogue would be to call a p-adic analytic space X p-adic

Brody hyperbolic if the only p-adic analytic maps $f: \mathbf{C}_p \to X$ are the constant maps, where $\mathbf{C}_p$ is the completion of the algebraic closure of $\mathbf{Q}_p$, the field of p-adic numbers. However, Vladimir Berkovich [**Ber**] has shown that there are no analytic maps defined on the entire p-adic plane mapping to a projective curve of genus ≥ 1. This differs from the case of the complex numbers since there are non-constant holomorphic maps from the entire complex plane into elliptic curves. Classically, one shows that any analytic map from the entire p-adic plane into $\mathbf{P}^1$ which omits two points is a constant. This is also different from the complex number case, where the map must omit three points before it is forced to be constant. Both of these differences can be explained philosophically by the failure of the exponential function to converge on the entire p-adic plane. In light of these differences, one should consider other possible definitions.

Tate has shown that elliptic curves with non-integral j-invariant can be uniformized. In other words, there is a p-adic analytic covering map from $\mathbf{C}_p^\times \to X$, where X is an elliptic curve with non-integral j-invariant and $\mathbf{C}_p^\times$ is the punctured p-adic plane. Furthermore, there are non-constant analytic maps from $\mathbf{C}_p^\times \to \mathbf{P}^1$ which omit two points; the function $f(z) = z$ is an example. Therefore, I might call a p-adic analytic space X "p-adic Brody hyperbolic," if every analytic map $f: \mathbf{C}_p^\times \to X$ must reduce to a constant. (It is useful to keep in mind that $\mathbf{C}_p^\times$ is simply connected as we will see later.) Note that in the complex case one can also define Brody hyperbolic as the condition that the only analytic maps $f: \mathbf{C}^\times \to X$ be the constant maps. This definition is clearly equivalent to the previous one because any map from $\mathbf{C}^\times$ can be composed with the exponential function to get a map from $\mathbf{C}$. For an algebraic variety X defined over $\mathbf{Q}$, one would like a definition of p-adic Brody hyperbolicity such that the p-adic analytification of X is p-adic Brody hyperbolic if and only if it is complex Brody hyperbolic. Unfortunately, with the above definition, an elliptic curve with good reduction is p-adic Brody hyperbolic. However, perhaps this is the best one can do because there is no open subset of $\mathbf{P}^1$ such that every analytic map $f: \Omega \to \mathbf{P}^1 - \{\text{three points}\}$ is constant, but such that there are non-constant analytic maps from Ω into an elliptic curve with good reduction. From now on, the adjective **p-adic Brody hyperbolic** will refer to the non-existence of non-constant analytic maps from $\mathbf{C}_p^\times$. Even though we do not have a definition of p-adic Brody hyperbolicity that would allow us to conjecture that a variety would be complex Brody hyperbolic if and only if it were p-adic Brody hyperbolic, studying the possible images of p-adic analytic maps in algebraic varieties is still an interesting subject and may someday give some insight toward Lang's conjecture.

In this work, I will elaborate on some of the above ideas. I begin in Chapter I with some classical results about functions on $\mathbf{C}_p^\times$, including a proof that every analytic map $f: \mathbf{C}_p^\times \to \mathbf{P}^1$ which misses three points must be a constant. In Chapter II, I review "p-adic Nevanlinna Theory" for meromorphic functions on $\mathbf{C}_p^\times$, as developed by Khoái and Quang [**KQ**], and Boutabaa [**Bo**], while I also show that their theory remains valid for meromorphic functions on $\mathbf{C}_p^\times$. As it turns out, a good analogy exists between the value distribution theory of p-adic analytic maps from $\mathbf{C}_p^\times$ into projective space and the classical theory of holomorphic curves in $\mathbf{P}^n$. The proofs of many of the theorems on holomorphic curves require only the Lemma on the Logarithmic Derivative from Nevanlinna theory, Borel's unit theorem, and projective linear algebra. Therefore, Chapter III begins by noting that the analogue to Borel's unit theorem on $\mathbf{C}_p^\times$ is a triviality and then gives the p-adic analogues to a number of theorems about holomorphic curves, most of which are contained in Chapter VII of [**L1**]. Included here is a p-adic analogue of Cartan's "Second Main Theorem" for holomorphic curves in $\mathbf{P}^n$.

Next I examine the images of p-adic analytic maps in projective varieties. A number of topological techniques and theorems, such as the theory of covering spaces, map liftings, and the uniformization theorem are useful in studying the behavior of holomorphic curves in complex analytic varieties. Until recently, one of the main drawbacks of p-adic analysis was that these topological techniques were not available. Now, thanks to Berkovich's [**Ber**] work, many of these topological techniques can be applied to the study of p-adic analysis as well. In Chapter IV, I explain Berkovich's theory of p-adic analytic spaces. I begin Chapter V with a review of the p-adic analytic Semi-Stable Reduction Theorem of Bosch and Lütkebohmert [**BL1,BL2**] together with the corresponding p-adic uniformization theorem for smooth projective algebraic curves. I then explain how Berkovich applies this to the study of p-adic analytic maps into algebraic curves, and I show that his proof shows that if X is a smooth projective curve of genus ≥ 2 or an elliptic curve with good reduction, then any analytic map $f\colon \mathbf{C}_p^\times \to X$ must be a constant. Similarly, I begin Chapter VI with a summary of the uniformization and reduction theory of Abelian varieties and analytic tori. I then use Berkovich's techniques to show that every p-adic analytic map from $\mathbf{C}_p$ into an Abelian variety must be a constant. Just as in the case of elliptic curves, there are no analytic maps from $\mathbf{C}_p^\times$ into Abelian varieties with good reduction, but there are non-constant maps in Abelian varieties with degenerate reduction. I finish this chapter by giving a proof of the following p-adic analogue to Bloch's Conjecture:

Theorem. *Let X be the analytification of a smooth projective variety with ample irregularity, which by definition means that* $\dim H^1(X, \mathcal{O}_X)$ *is larger than* $\dim X$. *Then, every analytic map $f\colon \mathbf{C}_p^\times \to X$ is contained in a proper algebraic subvariety.*

In the last chapter, I suggest an analogue for a different definition of hyperbolicity. In studying complex hyperbolic spaces, S. Kobayashi developed a semi-distance on complex analytic spaces, which is now called the Kobayashi semi-distance. The Kobayashi semi-distance on a complex analytic space satisfies all the properties of a distance except that two distinct points may be at distance zero from each other. This semi-distance gives another definition for a complex analytic space to be hyperbolic. Namely, one says that a complex analytic space X is **Kobayashi hyperbolic** if the Kobayashi semi-distance on X is in fact a legitimate distance. This is related to the notion of Brody hyperbolicity above by a theorem of Brody

[**Br**] which says, in its weakest form, that a compact complex analytic space is Brody hyperbolic if and only if it is Kobayashi hyperbolic. (Note there is an example which shows that the two notions are not equivalent for non-compact spaces.) In Chapter VII, I define a p-adic analogue to Kobayashi's semi-distance, and I prove some of its basic properties. Unfortunately, technical difficulties caused by the fact that $\mathbf{C}_p$ is not locally compact have prevented me from proving a p-adic analogue of Brody's theorem. However, I look at some special cases, like curves and some algebraic surfaces, to support the idea that the semi-distance I have defined fails to be a distance on a p-adic projective variety X only if there exists a non-constant analytic map from $\mathbf{C}_p$ into X. Now, recall that the existence of non-constant holomorphic maps from $\mathbf{C}$ into a variety X is supposed to be the result of X containing a subvariety which is the rational image of a group variety. However, in Chapter VI, I show that all p-adic analytic maps from $\mathbf{C}_p$ into Abelian varieties are constant. Therefore, the only groups that should give rise to non-constant analytic maps from $\mathbf{C}_p$ into X are additive groups. Thus, one might suspect that if X is a non-singular projective p-adic algebraic variety, then there is a non-constant analytic map from $\mathbf{C}_p$ into X if and only if there is a non-constant algebraic morphism from $\mathbf{P}^1$ into X. This is precisely the case in the examples given in Chapter VII, where in particular, I show that the p-adic analogue of the Kobayashi semi-distance on an Abelian variety is a legitimate distance. I conclude this final chapter with some open questions about how the Kobayashi semi-distance is related to the existence of rational curves.

The first three chapters of this work are for the most part elementary and assume only basic familiarity with hyperbolicity in the complex analytic case along with classical Nevanlinna theory. See [**L1**] and [**Hay**] for this background. The remaining chapters require a basic background in algebraic geometry and some familiarity with rigid analytic spaces. The standard references for these are [**Har**], [**BGR**] and [**F-P**].

Acknowledgement

I would like to thank my advisor, Professor Serge Lang, for suggesting these problems to me, for writing so many useful books, and for his constant advice and encouragement. I have learned a great deal studying Professor Vladimir Berkovich's inspiring book, and much of this work would not be possible without his theory. I thank Professors Vladimir Berkovich and G. D. Mostow for carefully reading this work and for their many helpful suggestions. I would like to thank Professors Robert Coleman, Jay Jorgenson, Shoshichi Kobayashi, Andrey Todorov, and Paul Vojta for their interest and their help. I am also grateful for the helpful comments that I received from Geoffrey Mess and Najmuddin Fakhruddin about elliptic fibrations. In addition, the constant support and encouragement of my fellow graduate students, the entire Yale mathematics faculty, my friends, and my mother have been a wonderful help. Finally, I would like to thank Yale University and the Alfred P. Sloan Foundation for their generous financial support.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

HYPERBOLIC p-ADIC ANALYTIC SPACES

WILLIAM CHERRY

CHAPTER I

p-Adic Analytic Functions

This chapter, following Amice [**Am**], collects the classical theory of analytic functions defined on $\mathbf{C}_p^\times$. Let $|\ |_p$ be the standard p-adic valuation on $\mathbf{Q}$, normalized so that $|p|_p = p^{-1}$. Let $\mathbf{Q}_p$ be the completion of $\mathbf{Q}$ with respect to this valuation, and let $\mathbf{C}_p$ be the completion of the algebraic closure of $\mathbf{Q}_p$. It is then a theorem that $\mathbf{C}_p$ is also algebraically closed. We still use $|\ |_p$ to denote the valuation on $\mathbf{C}_p$ which extends $|\ |_p$ on $\mathbf{Q}$. Let

$$v(z) = -\log_p |z|_p,$$

so in particular, $v(p) = 1$.

For $\mu, \nu \in \mathbf{R} \cup \{-\infty, \infty\}$, let

$$\begin{aligned} C(\mu,\nu) &= \{x \in \mathbf{C}_p : \mu < v(x) < \nu\} \quad \text{and} \\ C[\mu,\nu] &= \{x \in \mathbf{C}_p : \mu \le v(x) \le \nu\}. \end{aligned}$$

Let

$$\mathbf{C}_p^\times = C(-\infty,\infty) = \bigcup_{k=1}^{\infty} C(-k,k)$$

denote the units in $\mathbf{C}_p$, sometimes called the "punctured p-adic plane."

An **analytic function** f on $C(\mu,\nu)$ is defined to be a Laurent series

$$f(z) = \sum_{k\in\mathbf{Z}} a_k z^k$$

such that for $\mu < \gamma < \nu$,

$$\lim_{|k|\to\infty} |a_k|_p \gamma^k = 0,$$

or in other words, such that

$$\lim_{|k|\to\infty} (v(a_k) + k\gamma) = +\infty.$$

Later, a **meromorphic** function will be defined to be the quotient of two such series, but for now only analytic functions will be considered.

Given $f = \sum a_k z^k$, let $\mathrm{Conv}(f)$ denote the following subset of $\mathbf{R} \cup \{-\infty, \infty\}$:

$$\begin{aligned} -\infty \in \mathrm{Conv}(f) &\text{ if } a_k = 0 \text{ for all } k > 0, \\ \infty \in \mathrm{Conv}(f) &\text{ if } a_k = 0 \text{ for all } k < 0, \\ \gamma \in \mathrm{Conv}(f) &\text{ if } \lim_{|k| \to \infty} (v(a_k) + k\gamma) = \infty. \end{aligned}$$

A useful tool in the study of p-adic power series is the **valuation polygon.** Given an analytic function $f = \sum a_k z^k$, define

$$v(f, \mu) = \inf_{k \in \mathbf{Z}} \{v(a_k) + k\mu\}.$$

If $\mu \in \mathrm{Conv}(f)$, then $v(f, \mu) > -\infty$ because

$$\lim_{|k| \to \infty} v(a_k) + k\mu = \infty.$$

As a function of $\mu \in \mathrm{Conv}(f)$, $v(f, \mu)$ is piecewise linear, and the graph of $v(f, \mu)$, which is concave, is called the **valuation polygon.**

Let

$$\begin{aligned} k(f, \mu) &= \inf\{k \in \mathbf{Z} : v(a_k) + k\mu = v(f, \mu)\} \quad \text{and} \\ K(f, \mu) &= \sup\{k \in \mathbf{Z} : v(a_k) + k\mu = v(f, \mu)\}. \end{aligned}$$

Note that $k(f, \mu)$ and $K(f, \mu)$ are finite for $\mu \in \mathrm{Conv}(f)$.

In general, $k(f, \mu) \leq K(f, \mu)$, and if $k(f, \mu) \neq K(f, \mu)$ and $\mu \in \mathrm{Conv}(f)$, then μ is called a **critical point.** Note that if $f(z_0) = 0$, then $v(z_0)$ must be a critical point for f. The following proposition summarizes the important properties of the functions $k(f, \mu)$ and $K(f, \mu)$.

Proposition 1 *(4.2.6 in* [Am]*).*

(i) *The functions $k(f, \mu)$ and $K(f, \mu)$ are non-increasing for $\mu \in \mathrm{Conv}(f)$.*

(ii) *Let $\mu \in \mathrm{Conv}(f)$. There exists $\varepsilon_1 > 0$ such that if $\mu - \varepsilon_1 < \mu' < \mu$ and $\mu' \in \mathrm{Conv}(f)$ then $k(f, \mu') = K(f, \mu') = K(f, \mu)$, and there exists $\varepsilon_2 > 0$ such that if $\mu < \mu' < \mu + \varepsilon_2$ and $\mu' \in \mathrm{Conv}(f)$ then $K(f, \mu') = k(f, \mu') = k(f, \mu)$.*

(iii) *The function $k(f, \mu)$ is right semi-continuous and the function $K(f, \mu)$ is left semi-continuous.*

(iv) *Every compact interval in $\mathrm{Conv}(f)$ contains only finitely many critical points.*

The critical points are precisely those points where the slope of the valuation polygon changes. This next theorem, which appears in Amice as Theorem 4.4.4, shows that the number of zeros, counted with multiplicities, of f located at a critical point is precisely the change in slope of the valuation polygon at that critical point.

Theorem 2 (Hensel's Lemma). *Let f be a Laurent series, and let $\mu \in \mathrm{Conv}(f)$ be a critical point. Let $s = K(f,\mu) - k(f,\mu)$. There exists a unique pair (P,g) such that P is a polynomial of degree s, with $P(0) = 1$, $k(P,\mu) = 0$, $K(P,\mu) = s$, and such that g is a Laurent series with $\mu \in \mathrm{Conv}(g)$ and $f = Pg$. Furthermore, $\mathrm{Conv}(g) \supseteq \mathrm{Conv}(f)$ and $k(g,\mu) = K(g,\mu)$.*

This theorem is proved by successive approximation of P and g by repeatedly using the continuity of the Euclidean division algorithm for Laurent series.

Corollary 3. *A Laurent series with a critical point has a zero.*

Let $f(z) = \sum a_k z^k$ be a Laurent series which converges everywhere on $\mathbf{C}_p^\times$ such that infinitely many a_k are different from zero. If there are infinitely many $a_k \neq 0$ with $k < 0$, then

$$\lim_{\mu\to\infty} k(f,\mu) = \lim_{\mu\to\infty} K(f,\mu) = -\infty,$$

and similarly, if there are infinitely many $a_k \neq 0$ with $k > 0$, then

$$\lim_{\mu\to-\infty} k(f,\mu) = \lim_{\mu\to-\infty} K(f,\mu) = \infty.$$

Combining this observation with the above theorem, one sees that if f is an analytic function on $\mathbf{C}_p$ which is not a polynomial, then f has a zero, and in fact, infinitely many zeros. Similarly, an analytic function on $\mathbf{C}_p^\times$ which is not algebraic, meaning that f cannot be written as a polynomial in z and z^{-1}, has infinitely many zeros. This gives the following proposition.

Proposition 4. *Let $f\colon \mathbf{C}_p \to \mathbf{P}^1$ and $g\colon \mathbf{C}_p^\times \to \mathbf{P}^1$ be analytic maps (i.e. meromorphic functions). Assume there exist points x_1, x_2, y_1 and y_2 in $\mathbf{P}^1$ such that the sets $f^{-1}(x_1), f^{-1}(x_2), g^{-1}(y_1)$ and $g^{-1}(y_2)$ each contain at most a finite number of points. Then both f and g are algebraic maps (i.e. rational functions). Furthermore, if $f^{-1}(x_1)$ and $f^{-1}(x_2)$ are both empty, then f must be constant. Similarly, if the image of g omits three points in $\mathbf{P}^1$, then g must be constant.*

Proof: By composing f and g with appropriate projective linear transformations, we may assume that $x_1 = y_1 = 0$ and $x_2 = y_2 = \infty$. Hence, f and g are meromorphic functions with only finitely many zeros and poles. Thus we can multiply f and g by appropriate polynomials to get analytic functions without poles and only finitely many zeros. Call these new functions f and g also. Both f and g are then algebraic by the above observation. The function f is then a polynomial, and if f does not have a zero, it must be constant by the Fundamental Theorem of Algebra. Finally, if g omits three points, then by the above it is algebraic, but a non-constant algebraic function on $\mathbf{C}_p^\times$ can omit at most two points.

This same idea can be used to prove the analogue of what is know as the "Big Picard Theorem." Suppose that f is analytic on $C[0,\infty)$ – the punctured closed unit disc. Suppose further that f has an essential singularity at zero, which means that when f is expanded in a series

$$\sum_{k\in\mathbf{Z}} a_k z^k,$$

there are infinitely many $a_k \neq 0$ for $k < 0$. Hence, by the above observation, f has infinitely many zeros because its valuation polygon has infinitely many critical points. Therefore, one has

Proposition 5 (Big Picard). *If $f\colon C[0,\infty) \to \mathbf{P}^1$ is an analytic map omitting two points, then f can be extended to an analytic map from $C[0,\infty]$ into $\mathbf{P}^1$.*

CHAPTER II

Nevanlinna Theory on $\mathbf{C}_p^\times$

Recall that over the complex numbers there are two approaches to classical Nevanlinna theory. One approach is a differential geometric approach involving positive $(1,1)$ forms, Chern forms, etc. The other approach is purely algebraic. A. Boutabaa [**Bo**] works out Nevanlinna theory on $\mathbf{C}_p$, and his approach is very similar to the purely algebraic treatment of classical Nevanlinna theory given by Hayman [**Hay**]. We will see that the theory in the p-adic case is simpler than in the complex case in that we will not have the additional complications of the exceptional set. Furthermore, the bound on the error term in the p-adic case is better than that in the complex case. For other treatments of p-adic Nevanlinna theory, including some results toward a higher dimensional theory, see the papers by Hà Huy Khoái. As yet, there is no p-adic analogue to the classical geometric approach, and this is one of the main difficulties in developing a higher dimensional theory. In this chapter, I closely follow Boutabaa [**Bo**], except that I work on $\mathbf{C}_p^\times$ instead of $\mathbf{C}_p$. Because people are not accustomed to thinking about Nevanlinna theory on the punctured plane, all the details have been included.

1. Definitions and Notation. Again, an analytic function on $\mathbf{C}_p^\times$ is a Laurent series

$$f(z) = \sum_{k \in \mathbf{Z}} a_k z^k$$

such that

$$\lim_{|k| \to \infty} v(a_k) + k\mu = +\infty, \quad \text{for all } \mu \text{ in } \mathbf{R}.$$

A **meromorphic function** is the quotient of two analytic functions, and is also called an analytic map into $\mathbf{P}^1$.

For a meromorphic function $f = g/h$, with g and h analytic, define

$$v(f, \mu) = v(g, \mu) - v(h, \mu).$$

Note that this is well-defined since if g_1 and g_2 are two analytic functions then

$$v(g_1 g_2, \mu) = v(g_1, \mu) + v(g_2, \mu).$$

Since we are working on $\mathbf{C}_p^\times$, rather than $\mathbf{C}_p$, the "unit circle" will play the role of the origin in standard Nevanlinna theory, and $\mathbf{C}_p^\times$ will be exhausted by the annuli $C(-\rho, \rho)$, rather than by discs. For the remainder of this chapter ρ will be reserved for a non-negative real number, and μ will be used for an arbitrary real number. When comparing with the theory over $\mathbf{C}$, one should think of ρ as $\log r$, where r is the radius of a disc in $\mathbf{C}$. With the above in mind, the Nevanlinna functions are defined as follows.

Let

$$m_f(\rho) = v^+(1/f, \rho) + v^+(1/f, -\rho),$$

where $v^+(g, \mu) = \max\{0, v(g, \mu)\}$. The function m_f is called the **mean proximity** function and measures how often, on average, the value of f is close to infinity. Again, when comparing with the complex theory, one should think of $v^+(1/f, \rho)$ as

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi}.$$

Let

$$n_f(\rho) = \sum_{a \in C[-\rho, \rho]} \max\{0, -(\mathrm{ord}_a f)\}$$

denote the number of poles inside the closed annulus $C[-\rho, \rho]$, and let

$$N_f(\rho) = \sum_{a \in C[-\rho, \rho]} \max\{0, -(\mathrm{ord}_a f)\}(\rho - |v(a)|)$$

be the "integrated" **counting** function, which is a logarithmic average of how often f is equal to infinity. Finally, define the **height** $T_f(\rho) = m_f(\rho) + N_f(\rho)$. The height function is classically called the "characteristic function" and measures the growth of the function f.

Now, for any a in $\mathbf{C}_p$, define

$$\begin{aligned} m_{f,a}(\rho) &= m_{1/(f-a)}(\rho), \\ N_{f,a}(\rho) &= N_{1/(f-a)}(\rho), \\ T_{f,a}(\rho) &= T_{1/(f-a)}(\rho), \end{aligned} \qquad \text{and} \qquad \begin{aligned} m_{f,\infty}(\rho) &= m_f(\rho), \\ N_{f,\infty}(\rho) &= N_f(\rho), \\ T_{f,\infty}(\rho) &= T_f(\rho). \end{aligned}$$

2. Jensen's Formulae and the First Main Theorem. The following two theorems are the p-adic analogues of Jensen's formula. The first theorem is Jensen's formula for an analytic function on a disc, and the second is an annular version.

Theorem 2.1 (Jensen's Formula, Disc Version). *Let f be a meromorphic function on $\mathbf{D}_\lambda$, the (open) disc of radius $p^{-\lambda}$, given by the Laurent series*

$$f(z) = \sum a_k z^k.$$

Let k_0 be the smallest integer k such that $a_k \neq 0$. Then, for $\mu > \lambda$, one has

$$v(a_{k_0}) = v(f,\mu) + \sum_{\substack{a\ \in \mathbf{D}_\mu, \\ a\ \neq 0}} (\mathrm{ord}_a f)(v(a) - \mu) + (\mathrm{ord}_0 f)\mu.$$

Proof: It suffices to prove the formula when f is analytic and $f(0) \neq 0$. The proof is then similar to the proof of the next theorem, so is omitted here.

Theorem 2.2 (Jensen's Formula, Annular Version). *Let f be a meromorphic function on $C(-\lambda, \lambda)$. Then for $0 < \rho < \lambda$, one has*

$$2v(f,0) = v(f,-\rho) + v(f,\rho) + \sum_{a \in C(-\rho,\rho)} (\mathrm{ord}_a f)(\rho - |v(a)|),$$

or in other notation,

$$T_{1/f}(\rho) = T_f(\rho) + 2v(f,0).$$

Proof: It suffices to consider f analytic, in which case one needs to show that

$$2v(f,0) = v(f,-\rho) + v(f,\rho) + \sum_{a \in C(-\rho,\rho)} (\mathrm{ord}_a f)(\rho - |v(a)|).$$

This is a straight forward consequence of the properties of the valuation polygon as follows.

First, if f is multiplied by a scalar α then both the left hand side and right hand side are increased by $2v(\alpha)$, so we can multiply f by a scalar and assume that $v(f,0) = 0$. Furthermore, multiplying f by z^m leaves the left hand side alone, and since we are working on $C[-\rho, \rho]$ it does not affect the number of zeros or poles. Also, since

$$v(z^m f, \rho) = v(f,\rho) + m\rho \quad \text{and} \quad v(z^m f, -\rho) = v(f,-\rho) - m\rho,$$

the right hand side is also unchanged. Therefore after multiplying by a suitable power of z, we may assume $k(f,0) = 0$. Let $s = K(f,0)$, and hence $v(a_k) > 0$ for $k < 0$ and for $k > s$.

This means that the slope of the valuation polygon just to the right of zero is zero, and just to the left of zero is s. If μ is a critical point for the valuation polygon, then the change in slope at μ is equal to

$$\sum_{v(a)=\mu} (\mathrm{ord}_a f).$$

As there are only finitely many critical points with absolute value less than ρ, label the non-zero critical points as follows:

$$-\rho \le \mu_{-l} < \cdots < \mu_{-1} < 0 < \mu_1 < \cdots < \mu_m \le \rho.$$

First, we show

$$v(f,\rho) = -\sum_{0<v(a)\le\rho} (\mathrm{ord}_a f)(\rho - v(a))$$

by induction on m. If $m = 0$, then $v(f, 0) = 0$, and since the slope of the valuation polygon is always zero to the right of 0, $v(f,\rho) = 0$. Also, $(\mathrm{ord}_a f) = 0$ for all a such that $v(a) > 0$, so the above formula is true for $m = 0$.

By induction, we may assume that

$$v(f,\mu_m) = -\sum_{0<v(a)\le\mu_m} (\mathrm{ord}_a f)(\mu_m - v(a)).$$

The slope of the valuation polygon to the right of μ_m is

$$-\sum_{j=0}^{m} \sum_{v(a)=\mu_j} (\mathrm{ord}_a f).$$

Therefore,

$$\begin{aligned} v(f,\rho) &= -\sum_{0<v(a)\le\mu_m} (\mathrm{ord}_a f)(\mu_m - v(a)) - (\rho - \mu_m)\sum_{j=1}^{m}\sum_{v(a)=\mu_j} (\mathrm{ord}_a f) \\ &= -\sum_{0<v(a)\le\rho} (\mathrm{ord}_a f)(\rho - v(a)). \end{aligned}$$

Similarly,

$$v(f,-\rho) = -\sum_{-\rho\le v(a)\le 0} (\mathrm{ord}_a f)(\rho + v(a)),$$

keeping in mind that the slope of the valuation polygon just to the left of zero is given by

$$s = \sum_{v(a)=0} (\mathrm{ord}_a f).$$

Putting these two together gives the theorem.

Proposition 2.3. *One has the following:*

(1a)	$m_{f+g}(\rho) \le m_f(\rho) + m_g(\rho)$,	(1b)	$m_{fg}(\rho) \le m_f(\rho) + m_g(\rho)$,
(2a)	$N_{f+g}(\rho) \le N_f(\rho) + N_g(\rho)$,	(2b)	$N_{fg}(\rho) \le N_f(\rho) + N_g(\rho)$,
(3a)	$T_{f+g}(\rho) \le T_f(\rho) + T_g(\rho)$,	(3b)	$T_{fg}(\rho) \le T_f(\rho) + T_g(\rho)$.

Proof: Inequality (1b) follows from the fact that

$$v(fg,\mu) = v(f,\mu) + v(g,\mu), \quad \text{so} \quad v^+(fg,\mu) \le v^+(f,\mu) + v^+(g,\mu).$$

The inequality (1a) is a consequence of the fact that

$$|f(z) + g(z)|_p \le \max\{|f(z)|_p, |g(z)|_p\}.$$

The inequalities (2a) and (2b) are true because if z_0 is a pole for the sum or product of two functions then it must be a pole of at least one of the functions. Finally, the last two inequalities follow by adding the appropriate inequalities above.

Corollary 2.4. *For* $a \in \mathbf{C}_p^\times$,

$$T_{af}(\rho) = T_f(\rho) + O(1) \quad \textit{and} \quad T_{f-a}(\rho) = T_f(\rho) + O(1).$$

Corollary 2.5 (First Main Theorem). *For* $a \in \mathbf{C}_p$ *one has*

$$T_{f,a}(\rho) = T_f(\rho) + O(1).$$

Proof: Combine Corollary 2.4 with Jensen's formula.

The First Main Theorem says that, modulo a bounded term, $T_{f,a}$ does not depend on a and is therefore an intrinsic measure of the growth of the function. Furthermore, the First Main Theorem implies that if f does not assume the value a very often (i.e. $N_{f,a}$ grows slowly), then f must on average remain very close to a (i.e. $m_{f,a}$ grows quickly). This next proposition tells us that if T_f grows slowly enough, then f must be algebraic, or even constant.

Proposition 2.6. *Let f be a meromorphic function on* $\mathbf{C}_p^\times$. *Then*

(i) *f is constant if and only if* $T_f(\rho) = o(\rho)$.

(ii) *f is algebraic if and only if* $T_f(\rho) = O(\rho)$.

(iii) *f is non-constant if and only if there exists a constant $c \in \mathbf{R}$ and a number $A > 0$ such that $T_f(\rho) \ge \rho + c$ for $\rho > A$.*

Proof: To show (ii), first suppose $T_f(\rho) = O(\rho)$. This implies that there is a constant α such that

$$0 \le \frac{T_f(\rho)}{\rho} \le \alpha.$$

Now,

$$\begin{aligned} N_f(2\rho) &= \sum_{|v(a)| \le 2\rho} \max\{0, -(\mathrm{ord}_a f)\}(2\rho - |v(a)|) \\ &\ge \sum_{|v(a)| \le \rho} \max\{0, -(\mathrm{ord}_a f)\}(2\rho - |v(a)|) \\ &\ge \sum_{|v(a)| \le \rho} \max\{0, -(\mathrm{ord}_a f)\}\rho = n_f(\rho)\rho. \end{aligned}$$

This implies that

$$T_f(2\rho) \geq N_f(2\rho) \geq n_f(\rho)\rho \geq 0,$$

and hence

$$0 \leq n_f(\rho) \leq \frac{2T_f(2\rho)}{2\rho} \leq 2\alpha.$$

Therefore, f has a finite number of poles.

But, by Jensen's formula, one also has $T_{f,0}(\rho) = O(\rho)$, so f has only a finite number of zeros. Therefore, by Proposition 4 of Chapter I, f is algebraic.

Now, suppose that f is algebraic, and therefore

$$f(z) = z^s \frac{P(z)}{Q(z)},$$

where P and Q are polynomials without common factors such that $P(0) \neq 0$ and $Q(0) \neq 0$. Let a equal the degree of P and let b equal the degree of Q. For $\rho \gg 0$, one has

$$\begin{aligned} v(P,-\rho) &= -a\rho + O(1), & v(P,\rho) &= O(1), \\ v(Q,-\rho) &= -b\rho + O(1), & v(Q,\rho) &= O(1), \\ v(z^s,-\rho) &= -s\rho, & v(z^s,\rho) &= s\rho. \end{aligned}$$

Hence, for $\rho \gg 0$,

$$\begin{aligned} v(1/f,-\rho) &= (a-b+s)\rho + O(1) \quad \text{and} \\ v(1/f,\rho) &= -s\rho + O(1). \end{aligned}$$

Therefore,

$$m_f(\rho) = \begin{cases} O(1) & \text{if } b \geq a+s \text{ and } s \geq 0, \\ -s\rho + O(1) & \text{if } b \geq a+s \text{ and } s < 0, \\ (a-b+s)\rho + O(1) & \text{if } b < a+s \text{ and } s \geq 0, \\ (a-b)\rho + O(1) & \text{if } b < a+s \text{ and } s < 0, \end{cases}$$

for $\rho \gg 0$. Furthermore, for $\rho \gg 0$,

$$N_f(\rho) = b\rho + O(1).$$

Then,

$$T_f(\rho) = \begin{cases} b\rho + O(1) & \text{if } b \geq a+s \text{ and } s \geq 0, \\ (b-s)\rho + O(1) & \text{if } b \geq a+s \text{ and } s < 0, \\ (a+s)\rho + O(1) & \text{if } b < a+s \text{ and } s \geq 0, \\ a\rho + O(1) & \text{if } b < a+s \text{ and } s < 0, \end{cases}$$

for $\rho \gg 0$. Hence, $T_f(\rho) = O(\rho)$ and $T_f(\rho) = o(\rho)$ if and only if $a = b = s = 0$. Therefore, if f is algebraic and $T_f(\rho) = o(\rho)$ then f is constant.

To show (i), assume $f(z) = a$ is constant, then

$$T_f(\rho) = \max\{v(1/a), 0\} = o(\rho).$$

Now, suppose $T_f(\rho) = o(\rho)$. Suppose z_0 is a pole of f. This implies that

$$T_f(\rho) \geq N_f(\rho) \geq \rho - v(z_0),$$

which implies

$$\frac{T_f(\rho)}{\rho} \geq \frac{\rho - v(z_0)}{\rho},$$

which upon letting ρ tend to infinity implies $0 \geq 1$. Hence, f has no poles. Again, by Jensen's formula, $T_{f,0}(\rho) = o(\rho)$, so f does not have any zeros either. Therefore, f is algebraic by Proposition 4 of Chapter I. By what was said in the proof of part (ii), f must therefore be constant.

To show (iii), assume

$$f(z) = \frac{P(z)}{Q(z)}$$

is algebraic, so P and Q are polynomials without common zeros. One saw in the proof of part (ii) that

$$T_f(\rho) = \begin{cases} b\rho + O(1) & \text{if } a \leq b, \\ a\rho + O(1) & \text{if } a > b, \end{cases}$$

for $\rho \gg 0$, where a is the degree of P and b is the degree of Q. Hence, if f is non-constant, then a or b is ≥ 1 and we are done.

Now assume f is not algebraic. Then, again by Proposition 4 of Chapter I, f has either a zero or a pole. By Jensen's formula, we may assume that f has a pole, say z_0. Hence,

$$T_f(\rho) \geq N_f(\rho) \geq \rho - v(z_0),$$

and we are done.

3. Logarithmic Derivatives. The so-called Second Main Theorem of Nevanlinna theory examines in greater detail the respective contributions of the mean-proximity function m_f and the counting function N_f to the height function T_f. We proceed toward this theorem by first discussing logarithmic derivatives. This first lemma, The Lemma on the Logarithmic Derivative, says that for logarithmic derivatives, the mean-proximity function does not contribute very much to the height.

Lemma 3.1 (Logarithmic Derivative). *Let f be a meromorphic function on $\mathbf{C}_p^\times$, then*

$$m_{f'/f}(\rho) \leq \rho.$$

Proof: Suppose g is analytic on $\mathbf{C}_p^{\times}$. Write

$$g(z) = \sum_{k \in \mathbf{Z}} a_k z^k, \quad \text{so} \quad g'(z) = \sum_{k \neq 0} k a_k z^{k-1}.$$

Now,

$$\begin{aligned} v(g', \mu) &= \inf_{k \neq 0} \{v(k) + v(a_k) + (k-1)\mu\} \\ \text{[since } v(k) \geq 0] \qquad &\geq \inf_{k \neq 0} \{v(a_k) + (k-1)\mu\} \\ &= -\mu + \inf_{k \neq 0} \{v(a_k) + k\mu\} \\ &\geq -\mu + \inf_{k \in \mathbf{Z}} \{v(a_k) + k\mu\} = -\mu + v(g, \mu). \end{aligned}$$

This implies

$$v(g/g', \mu) = v(g, \mu) - v(g', \mu) \leq \mu.$$

Next, write $f = g/h$ where g and h are analytic without common zeros. Then,

$$\frac{f}{f'} = \frac{(g/g')(h/h')}{(h/h') - (g/g')},$$

and hence,

$$\begin{aligned} v(f/f', \mu) &\leq v(g/g', \mu) + v(h/h', \mu) - \min\{v(h/h', \mu), v(g/g', \mu)\} \\ &= \max\{v(g/g', \mu), v(h/h', \mu)\} \leq \mu. \end{aligned}$$

This implies that

$$m_{f'/f}(\rho) = v^+(f/f', \rho) + v^+(f/f', -\rho) \leq \rho + 0 = \rho,$$

concluding the proof the of the lemma.

Remark. Note the difference in the error term from the complex case, where $\log T_f$ is the dominant term on the right.

A couple of lemmas on the Nevanlinna functions of derivatives are still needed before the proof of the Second Main Theorem can be given.

Lemma 3.2. *Let $f = g/h$ where g and h are analytic on $\mathbf{C}_p^{\times}$ without common zeros. Let $G_0 = g$ and*

$$G_k = hG'_{k-1} - kh'G_{k-1}$$

for $k \geq 1$. Then,

$$f^{(l)} = \frac{G_l}{h^{l+1}}.$$

Furthermore, if z_0 is a zero of order e for h, then z_0 is a zero of order l(e-1) for G_l and hence a pole of order l+e for $f^{(l)}$.

Proof: The proof of this lemma follows immediately by induction on l.

Let

$$\bar{N}_f(\rho) = \sum_{a \in C(-\rho,\rho)} (\rho - |v(a)|).$$

The function $\bar{N}_f$ counts the poles of f without multiplicity.

Proposition 3.3. *Let f be a meromorphic function on $\mathbf{C}_p^\times$. Then*

$$N_{f^{(l)}}(\rho) = N_f(\rho) + l\bar{N}_f(\rho) \quad \textit{and}$$
$$T_{f^{(l)}}(\rho) \leq T_f(\rho) + l\bar{N}_f(\rho) + l\rho \leq (l+1)T_f(\rho) + l\rho.$$

Proof: Let $z_1, \ldots, z_q$ be the poles of f in $C(-\rho,\rho)$, and let m_i denote the order of the pole z_i. By Lemma 3.2, the poles of $f^{(l)}$ are also $z_1, \ldots, z_q$, and the orders of these poles are $m_1 + l, \ldots, m_q + l$. Therefore,

$$\begin{aligned} N_{f^{(l)}}(\rho) &= \sum (m_i + l)(\rho - |v(z_i)|) \\ &= \sum m_i(\rho - |v(z_i)|) + l \sum (\rho - |v(z_i)|) \\ &= N_f(\rho) + l\bar{N}_f(\rho), \end{aligned}$$

and hence the first inequality.

For the second inequality, letting

$$F = \frac{f^{(l)}}{f^{(l-1)}} \cdots \frac{f'}{f} f,$$

one has

$$\begin{aligned} T_{f^{(l)}}(\rho) &= N_{f^{(l)}}(\rho) + m_{f^{(l)}}(\rho) \\ &= N_f(\rho) + l\bar{N}_f(\rho) + m_F(\rho) \\ &\leq N_f(\rho) + l\bar{N}_f(\rho) + l\rho + m_f(\rho) \end{aligned}$$

by Lemma 3.1 and proposition 2.3. Hence,

$$\begin{aligned} T_{f^{(l)}}(\rho) &\leq T_f(\rho) + l\bar{N}_f(\rho) + l\rho \\ &\leq (l+1)T_f(\rho) + l\rho, \end{aligned}$$

and the proof is completed.

4. Ramification and the Second Main Theorem. Before starting the Second Main Theorem, the counting function for the ramification divisor needs to be defined. Let f be a meromorphic function on $\mathbf{C}_p^{\times}$ and write $f = g_1/g_0$, where g_0 and g_1 are analytic without common zeros on $\mathbf{C}_p^{\times}$. Let

$$W = W_f = \begin{vmatrix} g_0 & g_1 \\ g_0' & g_1' \end{vmatrix} = g_0 g_1' - g_0' g_1.$$

The zeros of W are the **ramification points** of f and the order of vanishing of W is the **order of ramification** of f. Thus, define

$$N_{f,\mathrm{Ram}}(\rho) = N_W(0,\rho) = N_{1/f'}(\rho) + 2N_f(\rho) - N_{f'}(\rho).$$

Now we are ready for the Second Main Theorem, which tells us two things. First, it is a theorem about the ramification points and how fast the associated counting function can grow relative to the height of f. Second, this theorem tells us that for most values of a, the counting function $N_{f,a}$ makes the dominant contribution to the height function. In other words, for most values of a, $m_{f,a}$ grows more slowly than does $T_{f,a}$. More precisely,

Theorem 4.1 (Second Main Theorem). *Let f be a meromorphic function on $\mathbf{C}_p^{\times}$, and let $a_1, \ldots, a_q$ be q distinct points in $\mathbf{P}^1$. Then,*

$$(q-2)T_f(\rho) - \sum_{j=1}^{q} N_{f,a_j}(\rho) + N_{f,\mathrm{Ram}}(\rho) \leq S(\rho),$$

where

$$S(\rho) = 2\rho + C,$$

and the constant C depends only on $q, v(f,0), v(f',0)$, and the points $a_1, \ldots, a_q$.

Remark. Again, notice that there is no $\log T_f$ in the error term as in the complex case; also note that there is no need for an exceptional set.

Proof: By the First Main Theorem, it suffices to prove

$$\sum_{j=1}^{q} m_{f,a_j}(\rho) + N_{f,\mathrm{Ram}}(\rho) \leq 2\,T_f(\rho) + S(\rho).$$

If one of the a_j is equal to ∞, say j_0, then

$$\sum_{j=1}^{q} m_{f,a_j}(\rho) = m_f(\rho) + \sum_{j \neq j_0} m_{f,a_j}(\rho).$$

Otherwise,

$$\sum_{j=1}^{q} m_{f,a_j}(\rho) \leq m_f(\rho) + \sum_{j=1}^{q} m_{f,a_j}(\rho).$$

Thus, we can assume that all of the a_j are in $\mathbf{C}_p$ if we can show

$$m_f(\rho) + \sum_{j=1}^{q} m_{f,a_j}(\rho) \leq 2T_f(\rho) + S(\rho).$$

Let

$$g = \sum_{j=1}^{q} \frac{1}{f - a_j}$$

and let $\delta \in \mathbf{R}_{\geq 0}$ such that $v(a_i - a_j) \leq \delta$ for all $i \neq j$.

First, we will show that

$$v^+(1/g, \rho) \geq \sum_{j=1}^{q} v^+(f - a_j, \rho) - q\delta.$$

This will be done in two cases.

Case 1: Suppose $v(f - a_j, \rho) > \delta$ for some j. In this case, for $i \neq j$, one has

$$v(f - a_i, \rho) = v(f - a_j + a_j - a_i, \rho) = v(a_j - a_i) \leq \delta.$$

Therefore,

$$\begin{aligned} v(1/g, \rho) &= -v(g, \rho) \\ &= -v(\sum 1/(f - a_i), \rho) \\ &= -v(1/(f - a_j), \rho) = v(f - a_j, \rho). \end{aligned}$$

This implies that

$$v^+(1/g, \rho) = v^+(f - a_j, \rho).$$

Therefore, since $v^+(f - a_i, \rho) \leq \delta$ for $i \neq j$, one has

$$\begin{aligned} v^+(1/g, \rho) &= v^+(f - a_j, \rho) \\ &\geq \sum_{i=1}^{q} v^+(f - a_i, \rho) - (q-1)\delta \\ &\geq \sum_{i=1}^{q} v^+(f - a_i, \rho) - q\delta. \end{aligned}$$

Case 2: Suppose $v(f-a_i,\rho)\le\delta$ for all i. Then,

$$v^+(f-a_i,\rho)-\delta\le 0 \quad \text{for all } i.$$

This implies that

$$v^+(1/g,\rho)\ge 0\ge\sum_{i=1}^{q}v^+(f-a_i,\rho)-q\delta.$$

In exactly the same way, one has

$$v^+(1/g,-\rho)\ge\sum_{j=1}^{q}v^+(f-a_i,-\rho)-q\delta,$$

and therefore,

$$m_g(\rho)\ge\sum_{j=1}^{q}m_{f,a_j}(\rho)-2q\delta. \qquad (*)$$

Furthermore since $g=(1/f)(f/f')f'g$ and by Jensen's formula, one gets

$$\begin{aligned} m_g(\rho)\le & m_{1/f}(\rho)+m_{f/f'}(\rho)+m_{f'g}(\rho)\\ = & [T_f(\rho)-N_{1/f}(\rho)+2v(f,0)]\\ & +[m_{f'/f}(\rho)+N_{f'/f}(\rho)-N_{f/f'}(\rho)+2v(f'/f,0)]\\ & +m_{f'g}(\rho)\end{aligned}$$

Rewriting $(*)$, one gets

$$m_f(\rho)+\sum_{j=1}^{q}m_{f,a_j}(\rho)\le m_f(\rho)+m_g(\rho)+2q\delta,$$

and combining this with the above, one has

$$\begin{aligned} m_f(\rho)+\sum_{j=1}^{q}m_{f,a_j}(\rho)\le\ & T_f(\rho)-N_{1/f}(\rho)+N_{f'/f}(\rho)-N_{f/f'}(\rho)\\ & +m_{f'/f}(\rho)+m_{f'g}(\rho)+2v(f',0)\\ & +T_f(\rho)-N_f(\rho)+2q\delta\\ =\ & 2T_f(\rho)-N_{1/f}(\rho)+N_{f'/f}(\rho)-N_{f/f'}(\rho)\\ & -N_f(\rho)+m_{f'/f}(\rho)+m_{f'g}(\rho)\\ & +2v(f',0)+2q\delta.\end{aligned}$$

Now,

$$N_{f'/f}(\rho) = \bar{N}_f(\rho) + \bar{N}_{1/f}(\rho) \quad \text{and} \quad N_{f/f'}(\rho) = N_{1/f'}(\rho) - [N_{1/f}(\rho) - \bar{N}_{1/f}(\rho)].$$

This, together with Lemma 3.3, implies that

$$\begin{aligned} N_{f'/f}(\rho) - N_{f/f'}(\rho) &= \bar{N}_f(\rho) - N_{1/f'}(\rho) + N_{1/f}(\rho) \\ &= N_{f'}(\rho) - N_f(\rho) + N_{1/f}(\rho) - N_{1/f'}(\rho). \end{aligned}$$

Now, notice that

$$N_{f,\mathrm{Ram}}(\rho) = 2N_f(\rho) + N_{1/f'}(\rho) - N_{f'}(\rho),$$

and therefore,

$$\begin{aligned} m_f(\rho) &+ \sum_{j=1}^{q} m_{f,a_j}(\rho) + N_{f,\mathrm{Ram}}(\rho) \\ &\leq 2T_f(\rho) + m_{f'/f}(\rho) + m_{f'g}(\rho) + 2q\delta + 2v(f',0). \end{aligned}$$

However,

$$f'g = \frac{F'}{F} \quad \text{where} \quad F = (f-a_1)\cdots(f-a_q).$$

Therefore by Lemma 3.1, the right hand side of the above inequality is less than or equal to

$$2\rho + 2q\delta + 2v(f',0),$$

and the proof of the theorem is complete.

Remark. Note that Theorem 4.1 can be used to give another (although more complicated) proof of Proposition 4 of Chapter I. Indeed, if f omits three points, then Theorem 4.1 implies that

$$T_f(\rho) = O(\rho),$$

which implies that f is algebraic by Proposition 2.6, and hence constant.

CHAPTER III

Analytic Maps from $\mathbf{C}_p^\times$ into $\mathbf{P}^n$

In this chapter, I follow Chapter VII of [**L1**] to give some more examples of spaces which are p-adic Brody hyperbolic. The proofs of most of these results use only the Lemma on the Logarithmic Derivative of the previous chapter, the analogue of Borel's unit theorem given below, and projective linear algebra. Therefore, I have been able to use the same proofs given in [**L1**] verbatim, but I have repeated them here for the convenience of the reader. The p-adic results here are easier than their complex analytic counterparts because there are no non-algebraic functions on $\mathbf{C}_p^\times$ which are also units. In fact, by Proposition 4 of Chapter I, the units on $\mathbf{C}_p^\times$ are all of the form cz^d.

1. Analytic Maps Missing Hyperplanes. Because all the units on $\mathbf{C}_p^\times$ are of the form cz^d, Borel's Theorem on $\mathbf{C}_p^\times$ is a triviality.

Theorem 1.1 (Borel's Theorem). *Let $h_0, \ldots, h_n$ be units on $\mathbf{C}_p^\times$. Suppose*

$$h_0 + \cdots + h_n = 0.$$

Define an equivalence relation on $\{0, \ldots, n\}$ by $i \sim j$ if there exists a constant c such that $h_i = ch_j$. Let $\{S\}$ be the partition of $\{0, \ldots, n\}$ into equivalence classes. Then for each equivalence class S,

$$\sum_{i \in S} h_i = 0.$$

If $n \leq 2$, then there is only one equivalence class.

Proof: This is an immediate consequence of the linear independence of

$$\{z^d : d \in \mathbf{Z}\}.$$

Once this is established, the exact proofs given in [**L1**] for the complex case prove the p-adic versions of the next two theorems, which show that analytic maps from $\mathbf{C}_p^\times$ into $\mathbf{P}^n$ omitting sufficiently many hyperplanes are degenerate in the sense that their images are contained in proper linear subspaces of $\mathbf{P}^n$.

Theorem 1.2 (Bloch-Cartan). *Let $f\colon \mathbf{C}_p^\times \to \mathbf{P}^n$ be a non-constant analytic map with $n \geq 2$. Let $H_0, \ldots, H_{n+1}$ be $n+2$ hyperplanes in general position. If the image of f lies in the complement of $H_0 \cup \cdots \cup H_{n+1}$, then it lies in some diagonal hyperplane.*

Proof: Choose projective coordinates $x_0, \ldots, x_n$ for $\mathbf{P}^n$ so that the hyperplanes are given by

$$x_0 = 0, \quad \ldots, \quad x_n = 0, \quad x_0 + \cdots + x_n = 0.$$

Let $(f_0, \ldots, f_n)$ be a coordinate representation for f such that the f_i are analytic on $\mathbf{C}_p^{\times}$ and without common zeros. By the hypotheses in the theorem, $f_i(z) \neq 0$ for all i and all z. Furthermore,

$$f_0(z) + \cdots + f_n(z) \neq 0 \qquad \text{for all } z.$$

Let

$$h_i = \frac{f_i}{f_0 + \cdots + f_n}.$$

From above, each h_i is analytic and non-zero on $\mathbf{C}_p^{\times}$, and

$$h_0 + \cdots + h_n = 1.$$

By Theorem 1.1 and the assumption that f is non-constant, there exists a subset S of $\{0, \ldots, n\}$ such that

$$\sum_{i \in S} h_i = 0,$$

and each h_i for i in S is non-constant. The linear relation

$$\sum_{i \in S} x_i = 0$$

determines the diagonal hyperplane that contains the image of f.

Theorem 1.3 (Fujimoto and Green). *Let $f: \mathbf{C}_p^{\times} \to \mathbf{P}^n$ be an analytic map. Assume that the image of f lies in the complement of $n + r$ hyperplanes in general position. Then, this image is contained in a linear subspace of dimension $\leq [n/r]$.*

Proof: Let $x_0, \ldots, x_n$ be projective coordinates for $\mathbf{P}^n$ and let

$$H_1(x), \ldots, H_{n+r}(x)$$

be the linear forms defining the hyperplanes $H_k(x) = 0$. By assumption, any $n+1$ of these forms are linearly independent and any $n + 2$ satisfy a linear relation with coefficients which are all different from zero because if one of the coefficients were zero, then that would mean $n + 1$ of the forms were linearly dependent. Let $f_0, \ldots, f_n$ be coordinate representatives for f, where f_i are analytic without common zeros. Let $h_k = H_k \circ f$ for $k = 1, \ldots, n + r$. Then, h_k is a unit on $\mathbf{C}_p^{\times}$ by the hypotheses of the theorem. Let

$$\{1, \ldots, n + r\} = S_1 \cup \cdots \cup S_q$$

be a partition of $\{1, \ldots, n+r\}$ into equivalence classes as in Theorem 1.1.

Now, the complement of each equivalence class contains at most n elements. To see this, assume that the complement of some S_i contains at least $n+1$ elements. Choose $n+1$ elements not in S_i and one element in S_i to get a set of $n+2$ elements, which we will denote by J. As noted above, since J consists of $n+2$ elements, there exist coefficients $a_j \neq 0$ such that

$$\sum_{j \in J} a_j H_j = 0.$$

This implies that

$$\sum_{j \in J} a_j h_j = 0.$$

Let j_0 be the unique element in $S_i \cap J$. Then, Theorem 1.1 would imply that $h_{j_0} = 0$, and this is a contradiction. Therefore, the complement of each equivalence class contains at most n elements, or in other words, each equivalence class contains at least r elements. This implies that $qr \leq n+r$ since a set of $n+r$ elements has been broken up into q disjoint subsets, each of which has at least r elements.

Let T be any subset of $\{1, \ldots, n+r\}$ consisting of $n+1$ elements. Write

$$T = T_1 \cup \cdots \cup T_q \qquad \text{where} \quad T_k = T \cap S_k.$$

Let t_k denote the cardinality of the set T_k. Fixing an x_i for each set T_k, one gets $t_k - 1$ linear relations $x_j - c_j x_i = 0$ coming from the fact that $h_j - c_j h_i = 0$ for some c_j since i and j are in the same equivalence class. Note that the image of f is contained in the subspace defined by these relations. Furthermore, since the set of H_j for j in T are linearly independent, one gets that the image of f is contained in a linear subspace defined by the vanishing of at least

$$\begin{aligned} t_1 - 1 + \cdots + t_q - 1 &\geq n + 1 - q \\ &\geq n + 1 - \frac{n+r}{r} = n - \frac{n}{r} \end{aligned}$$

linearly independent linear forms, and hence the theorem.

Corollary 1.4. *If $f\colon \mathbf{C}_p^\times \to \mathbf{P}^n$ is analytic and the image of f lies in the complement of $2n+1$ hyperplanes in general position, then f is constant.*

Hence, any subset of $\mathbf{P}^n$ contained in the complement of $2n+1$ hyperplanes in general position is p-adic Brody hyperbolic.

In a recent preprint, [**Ru**], Min Ru has given a linear algebra condition on a set of hyperplanes $\mathcal{H}$ so that $\mathbf{P}^n - |\mathcal{H}|$ is Brody hyperbolic if and only if $\mathcal{H}$ satisfies this linear algebra condition. The rest of this section describes this result and is due to Min Ru.

I begin with

Theorem 1.5 (Ru). *Let* $f\colon \mathbf{C}_p^{\times} \to \mathbf{P}^n$ *be an analytic map. If the image of* f *omits at least three distinct hyperplanes in* $\mathbf{P}^n$ *which are linearly dependent, then the image of* f *must be contained in a proper linear subspace of* $\mathbf{P}^n$.

Proof: Let $H_1, \ldots, H_q$, $q \geq 3$ be the distinct linearly dependent hyperplanes that the image of f omits. By my usual abuse of notation, let $H_1(x), \ldots, H_q(x)$ also denote the linear forms defining the hyperplanes.

By the linear dependence assumption, there exist non-zero constants c_i such that

$$\sum_{i=1}^{q} c_i H_i(x) = 0.$$

Without loss of generality, by shrinking the set of hyperplanes, we may assume that q is the smallest integer such that we have such a relation (i.e. $c_i \neq 0$ for all i). Because the hyperplanes are distinct, we still have $q \geq 3$. Now,

$$\sum_{i=1}^{q} c_i H_i(f) \equiv 0,$$

so

$$\sum_{i=2}^{q} \frac{-c_i H_i(f)}{c_1 H_1(f)} \equiv 1.$$

Therefore by Theorem 1.1, Borel's Theorem, there exist constants d_i, not all zero, such that

$$\sum_{i=2}^{q} d_i \frac{c_i H_i(f)}{c_1 H_1(f)} = 0,$$

and hence

$$\sum_{i=2}^{q} d_i c_i H_i(f) = 0.$$

Therefore, the image of f is contained in the subspace defined by

$$\sum_{i=2}^{q} (d_i c_i) H_i(x) = 0,$$

and this subspace is a proper subspace by the assumption that q was minimal.

Thus, one sees that when the image of an analytic map omits the image of linearly dependent hyperplanes, it is degenerate. The idea is to define a linear algebra condition, so that the map must be degenerate in every subspace, and hence constant. Ru's definitions are as follows.

Let $\mathcal{H}$ be a set of hyperplanes in $\mathbf{P}^n$, and let V be a linear subspace of $\mathbf{P}^n$. Then, the subspace V is called **$\mathcal{H}$-admissible** if V is not contained in any of the hyperplanes in $\mathcal{H}$. The set of hyperplanes $\mathcal{H}$ is called **perfect** if for every $\mathcal{H}$-admissible subspace V of positive dimension, $\mathcal{H} \cap V$ contains at least three distinct hyperplanes of V which are linearly dependent. With these definitions and applying an induction argument using Theorem 1.5, Ru gets

Theorem 1.6 (Ru). *If $\mathcal{H}$ is a set of hyperplanes in $\mathbf{P}^n$, then $\mathbf{P}^n - |\mathcal{H}|$ is p-adic Brody hyperbolic if and only if $\mathcal{H}$ is perfect.*

Proof: First we show that $\mathcal{H}$ perfect implies that $\mathbf{P}^n - |\mathcal{H}|$ is Brody hyperbolic. Let $\mathcal{H}$ be a perfect set of hyperplanes in $\mathbf{P}^n$ and let

$$f\colon \mathbf{C}_p^\times \to \mathbf{P}^n - |\mathcal{H}|$$

be an analytic map. Since $\mathcal{H}$ is perfect, $\mathcal{H}$ contains at least three distinct hyperplanes which are linearly dependent. Theorem 1.5 then implies that the image of f is contained in a proper linear subspace W of $\mathbf{P}^n$. Because the image of f is contained in W and omits $|\mathcal{H}|$, the subspace W is $\mathcal{H}$-admissible. Because $\mathcal{H}$ is perfect, we can use induction on W and $\mathcal{H} \cap W$ to conclude that f is constant.

Now let $\mathcal{H}$ be a set of distinct hyperplanes in $\mathbf{P}^n$ which is not perfect. Because $\mathcal{H}$ is not perfect, there exists a positive dimensional subspace V of $\mathbf{P}^n$ which is $\mathcal{H}$-admissible, but such that $\mathcal{H} \cap V$ does not contain three distinct linearly dependent hyperplanes. Without loss of generality, we may assume that $V = \mathbf{P}^n$. Let $\mathcal{H} = \{H_0, \ldots, H_q\}$. Now, $q \leq n$ and $H_0, \ldots, H_q$ are linearly independent. Hence, we may assume that $H_0, \ldots, H_q$ are the first q coordinate hyperplanes. So, let

$$f\colon \mathbf{C}_p^\times \to \mathbf{P}^n - |\mathcal{H}|$$

be given by

$$z \mapsto (1, z, z, \ldots, z).$$

Then, f is not constant, so $\mathbf{P}^n - |\mathcal{H}|$ is not Brody hyperbolic.

The significance of Ru's work lies not so much in the above theorem, but in the following linear algebra lemma which allows one to decide whether or not a set of hyperplanes is "perfect."

Lemma 1.7 (Ru). *Let $\mathcal{H}$ be a set of hyperplanes in $\mathbf{P}^n$. Let $\mathcal{L}$ denote the corresponding set of linear forms. The set $\mathcal{H}$ is perfect if and only if*

$$\dim(\mathcal{L}) = n + 1,$$

and for each proper non-empty subset $\mathcal{L}_1$ of $\mathcal{L}$,

$$(\mathcal{L}_1) \cap (\mathcal{L} \setminus \mathcal{L}_1) \cap \mathcal{L} \neq \emptyset.$$

Before giving the proof, I note, as does Ru, that this gives an alternate proof to Corollary 1.4. Ru also gives the following example of a perfect set of hyperplanes which are not in general position.

Example 1.8. Let

$$\mathcal{L} = \{x_0, x_1, x_2, x_0 + x_1, x_0 + x_1 + x_2\}$$

be the set of linear forms associated to five distinct hyperplanes $\mathcal{H}$ in $\mathbf{P}^2$. Then, one can check that $\mathcal{L}$ satisfies the conditions of Lemma 1.7, so $\mathcal{H}$ is perfect even though the hyperplanes in $\mathcal{H}$ are not in general position. Theorem 1.6 then implies that $\mathbf{P}^2 - |\mathcal{H}|$ is p-adic Brody hyperbolic.

Before starting the proof of Lemma 1.7, we need one more lemma.

Lemma 1.9 (Ru). *Let $\mathcal{H}$ be a set of distinct hyperplanes on $\mathbf{P}^n$, and let $\mathcal{L}$ be the associated set of linear forms defining $\mathcal{H}$. Assume that either*

(*a*) $$\dim(\mathcal{L}) < n + 1$$

or

(*b*) $$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \quad \textit{where} \quad (\mathcal{L}_1) \cap (\mathcal{L}_2) = (0).$$

Then, there exists a positive dimensional $\mathcal{H}$-admissible linear subspace V of $\mathbf{P}^n$ such that $\mathcal{H} \cap V$ does not contain at least three distinct hyperplanes which are linearly dependent.

Proof: We split the proof into two cases depending on whether $\mathcal{L}$ satisfies assumption (a) or (b).

Case (a): Let $\mathcal{L}^*$ be a maximal subset of $\mathcal{L}$ such that all of the linear forms in $\mathcal{L}^*$ are linearly independent. Let $\mathcal{L}^* = \{L_1, \ldots, L_r\}$. If $r = \dim(\mathcal{L}) = 1$, then let $W = (0)$. If $r > 1$, because there are only finitely many elements in $\mathcal{L}$ and because all of the elements in $\mathcal{L}^*$ are linearly independent, there exist non-zero constants $c_2, \ldots, c_r$ such that

$$(L_2 - c_2 L_1, \ldots, L_r - c_r L_1) \cap \mathcal{L} = \emptyset.$$

In this case, put $W = (L_2 - c_2 L_1, \ldots, L_r - c_r L_1)$. Now, let

$$V = \{x \in \mathbf{P}^n : L(x) = 0 \text{ for all } L \in W\}.$$

Then, V has dimension

$$n - \dim(W) = n - (\dim(\mathcal{L}^*) - 1) = n + 1 - \dim(\mathcal{L}) > 0$$

by assumption (a). Clearly V is $\mathcal{H}$-admissible. The proof in this case is completed by noticing that the only hyperplane in $\mathcal{H} \cap V$ is the hyperplane $H_1 \cap V$ defined by the linear form L_1.

Case (b): As above, let $\mathcal{L}_i^*$ be a maximal subset of $\mathcal{L}_i$ such that all of the linear forms in $\mathcal{L}_i^*$ are linearly independent. Note that assumption (b) implies that all the forms in $\mathcal{L}^* = \mathcal{L}_1^* + \mathcal{L}_2^*$ are linearly independent. Let $\mathcal{L}_i = \{L_{i,1}, \ldots, L_{i,r_i}\}$. If $r_i = 1$, let $W_i = (0)$, and if $r_i > 1$, let

$$W_i = (L_{i,2} - c_{i,2}L_{i,1}, \ldots, L_{i,r_i} - c_{i,r_i}L_{i,1}),$$

where the non-zero constants $c_{i,j}$ are chosen so that

$$(W_1 + W_2) \cap \mathcal{L} = \emptyset.$$

This can be done because $\mathcal{L}$ is finite and the linear forms in $\mathcal{L}^*$ are linearly independent. Let $W = W_1 + W_2$, and as above let

$$V = \{x \in \mathbf{P}^n : L(x) = 0 \text{ for all } L \in W\}.$$

Then, V has dimension

$$n - \dim(W) = n - (\dim(\mathcal{L}^*) - 2) = n + 2 - \dim(\mathcal{L}) > 0.$$

Again, V is clearly $\mathcal{H}$-admissible. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the subsets of $\mathcal{H}$ corresponding to $\mathcal{L}_1$ and $\mathcal{L}_2$. Also, let $H_{i,1}$ denote the hyperplane defined by $L_{i,1}$. Then,

$$\mathcal{H} \cap V = \{H_{1,1} \cap V, H_{2,1} \cap V\},$$

so $\mathcal{H} \cap V$ does not contain three distinct hyperplanes of V, much less three distinct linearly dependent hyperplanes. This completes the proof of Lemma 1.9.

Proof of Lemma 1.7 : Recall that $\mathcal{H}$ is a set of hyperplanes in $\mathbf{P}^n$ and $\mathcal{L}$ is the set of linear forms associated to $\mathcal{H}$.

First assume that $\dim(\mathcal{L}) = n + 1$ and that for every subset $\mathcal{L}_1$ of $\mathcal{L}$, one has

$$(*) \qquad (\mathcal{L}_1) \cap (\mathcal{L} \setminus \mathcal{L}_1) \cap \mathcal{L} \neq \emptyset.$$

We have to show that $\mathcal{H}$ is perfect, so let V be an $\mathcal{H}$-admissible subspace of positive dimension. Denote by $\mathcal{L}'$ a maximal subset of $\mathcal{L}$ such that the hyperplanes in V defined by the forms in $\mathcal{L}'$ are all distinct. Because $\dim(\mathcal{L}) = n + 1$, the set $\mathcal{L}'$ contains at least two elements. So, let $\mathcal{L}_1'$ be a proper, non-empty subset of $\mathcal{L}'$. Let $\mathcal{L}_1$ be the largest subset of $\mathcal{L}$ such that the set of hyperplanes of V defined by the forms in $\mathcal{L}_1$ is the same as the set of hyperplanes in V defined by the forms in $\mathcal{L}_1'$. By $(*)$, we get

$$(\mathcal{L}_1') \cap (\mathcal{L}' \setminus \mathcal{L}_1') \cap \mathcal{L} = (\mathcal{L}_1) \cap (\mathcal{L} \setminus \mathcal{L}_1) \cap \mathcal{L} \neq \emptyset.$$

Therefore, there are linear forms $L_1, \ldots, L_p$ in $\mathcal{L}_1'$, and $L_{p+1}, \ldots, L_q$ in $\mathcal{L}' \setminus \mathcal{L}_1'$, with $q > p \geq 1$ together with non-zero constants α_i such that

$$\sum_{i=1}^{p} \alpha_i L_i = - \sum_{i=p+1}^{q} \alpha_i L_i \neq 0.$$

Therefore,

$$\sum_{i=1}^{q} \alpha_i L_i = 0,$$

with all the $\alpha_i \neq 0$. Because all of the hyperplanes in V defined by the linear forms in $\mathcal{L}'$ are distinct, q must be at least three. By the definition of $\mathcal{L}'$, all the hyperplanes in $\mathcal{H}$ coming from the forms L_i above are distinct in V, so we have at least three distinct linearly dependent hyperplanes in $\mathcal{H} \cap V$. Hence, $\mathcal{H}$ is perfect.

Now for the converse. If $\dim(\mathcal{L}) < n+1$, then Lemma 1.9 immediately implies that $\mathcal{H}$ is not perfect. So, we may assume that $\dim(\mathcal{L}) = n+1$, but that there is a non-empty, proper subset $\mathcal{L}_1$ of $\mathcal{L}$ such that

$$(**) \qquad (\mathcal{L}_1) \cap (\mathcal{L} \setminus \mathcal{L}_1) \cap \mathcal{L} = \emptyset.$$

Let

$$V = \{x \in \mathbf{P}^n : L(x) = 0 \text{ for all } L \text{ in } (\mathcal{L}_1) \cap (\mathcal{L} \setminus \mathcal{L}_1)\}.$$

The fact that $(**)$ holds implies that V is $\mathcal{H}$-admissible. The fact that $\mathcal{L}_1$ is a proper subset of $\mathcal{L}$ and $(**)$ imply that

$$\dim((\mathcal{L}_1) \cap (\mathcal{L} \setminus \mathcal{L}_1)) < \dim(\mathcal{L}_1) < \dim(\mathcal{L}) = n+1,$$

and therefore V is positive dimensional. Hence, Lemma 1.9 again implies that $\mathcal{H}$ is not perfect, and the proof of Lemma 1.7 is complete.

2. The Height of a Map into $\mathbf{P}^n$. Before proving any more theorems, we need to discuss the height of a map into projective space. Let $f\colon \mathbf{C}_p^\times \to \mathbf{P}^n$ be an analytic map with meromorphic coordinate functions $(f_0, \ldots, f_n)$. Define the **Cartan-Nevanlinna height**, or simply the **height**, of the map into projective space by

$$T_f(\rho) = \max_i v(1/f_i, \rho) + \max_i v(1/f_i, -\rho) - 2 \max_i v(1/f_i, 0) + \sum_{a \in C(-\rho,\rho)} \max_i \{\mathrm{ord}_a(1/f_i)\}(\rho - |v(a)|).$$

By Jensen's formula (Theorem II.2.2), T_f is independent of the choice of coordinate representatives for f. However, T_f does depend on the choice of projective coordinates for $\mathbf{P}^n$, but in light of the next proposition, this fact is omitted from the notation.

Proposition 2.1. *If* $A: \mathbf{P}^n \to \mathbf{P}^n$ *is an invertible projective linear transformation, then*

$$T_{A\circ f}(\rho) = T_f(\rho) + O(1),$$

where the $O(1)$ *term depends on* A *but not on* f. *Thus, the height* T_f *is independent (modulo a bounded term) of the choice of projective coordinates on* $\mathbf{P}^n$.

Proof: It suffices to assume that $f_1, \ldots, f_n$ are analytic, and then the result is an easy consequence of the fact that

$$v(a_{i0}f_0 + \cdots + a_{in}f_n, \mu) \geq v(a_{i0}) + v(f_0, \mu) + \cdots + v(a_{in}) + v(f_n, \mu),$$

where the last sum on the right omits the terms with $a_{ij} = 0$.

As a direct consequence of Proposition II.2.6, one then gets

Proposition 2.2. *Let* $f = (f_0, \ldots, f_n): \mathbf{C}_p^\times \to \mathbf{P}^n$. *Then,* $T_f(\rho) = o(\rho)$ *if and only if* f *is constant, and* $T_f(\rho) = O(\rho)$ *if and only if every* f_j *is algebraic.*

Wronskians. Given n functions $f_1, \ldots, f_n$, let $W(f) = W(f_1, \ldots, f_n)$ denote the Wronskian of these functions. Let

$$L(f_1, \ldots, f_n) = \frac{W(f_1, \ldots, f_n)}{f_1 \cdots f_n}$$

denote the logarithmic Wronskian, and let

$$L_i(f_1, \ldots, f_n) = L(f_1, \ldots, f_{i-1}, 1, f_{i+1}, \ldots, f_n)$$

be the logarithmic Wronskian with the i-th function replaced by 1. Two properties of logarithmic Wronskians will be useful to us. First, logarithmic Wronskians are invariant under scaling by functions. That is to say,

$$L(gf_1, \ldots, gf_n) = L(f_1, \ldots, f_n).$$

Second, each term in the expansion of the logarithmic Wronskian is a product of logarithmic derivatives, and therefore we will be able to apply the Lemma on the Logarithmic Derivative, Lemma II.3.1, to these Wronskians.

3. Fermat Varieties. In this section, I give p-adic versions of theorems due to Brody and Green about holomorphic curves in Fermat varieties and perturbations of Fermat varieties. The first theorem, due to Green, says that when the degree of a Fermat hypersurface in $\mathbf{P}^n$ is sufficiently large relative to n, then the only analytic maps from $\mathbf{C}_p^{\times}$ into the hypersurface are the "obvious" maps, all of whose images are contained in proper linear subspaces of $\mathbf{P}^n$. The following proposition about algebraic maps into Fermat hypersurfaces is extracted from [**L1**] and will be used in the proof of Green's theorem.

Proposition 3.1. *Let $f = (f_0, \ldots, f_n) \colon \mathbf{C}_p \to \mathbf{P}^n$ be an algebraic map such that the f_i are polynomials without common factors and such that*

$$f_0^d + \cdots + f_n^d = 0.$$

Assume that the functions in any proper subset of

$$\{f_i^d : i = 0, \ldots n\}$$

are linearly independent. Then, d is less than n^2.

Proof: Assume that f is not constant, so that at least one of the f_j, say f_0, is not identically zero. For $i = 1 \ldots n$, let $g_i = f_i/f_0$. Let $(L_0, \ldots, L_n)$ be the logarithmic Wronskians defined by

$$L_0 = L(g_1^d, \ldots, g_n^d) \quad \text{and} \quad L_i = L_i(g_1^d, \ldots, g_n^d) \text{ for } i = 1 \ldots n.$$

Note that since logarithmic Wronskians are invariant under scalings, we also have

$$L_0 = L(f_1^d, \ldots, f_n^d) \quad \text{and} \quad L_i = L(f_1^d, \ldots, f_{i-1}^d, f_0^d, f_{i+1}^d, \ldots, f_n^d) \text{ for } i = 1, \ldots, n.$$

By the linear independence assumption on proper subsets of the f_i^d, none of the L_i are identically zero. Again by assumption, we have

$$g_1^d + \cdots + g_n^d = -1.$$

This gives rise to the system of equations:

$$\begin{array}{ccccccc} g_1^d & + & \cdots & + & g_n^d & = & -1 \\ \dfrac{(g_1^d)'}{g_1^d} g_1^d & + & \cdots & + & \dfrac{(g_n^d)'}{g_n^d} g_n^d & = & 0 \\ & & \vdots & & & & \\ \dfrac{(g_1^d)^{(n-1)}}{g_1^d} g_1^d & + & \cdots & + & \dfrac{(g_n^d)^{(n-1)}}{g_n^d} g_n^d & = & 0. \end{array}$$

Cramer's Rule then tells us that

$$g_i^d = -\frac{L_i}{L_0} \quad \text{for} \quad i = 1, \ldots n.$$

Therefore, $f^d = (f_0^d, \ldots, f_n^d)$, $g^d = (1, g_1^d, \ldots, g_n^d)$, and $\Lambda = (L_0, \ldots, L_n)$ all represent the same map into projective space.

The idea is to now compare the degrees in the map f^d to the degrees in the map Λ. So, let e denote the maximum degree of the polynomials f_i, and we begin to take a closer look at the L_i. Note that if P is a polynomial of degree e, then

$$(P^d)^{(k)} = P^{d-1}P_1 + \cdots + P^{d-k}P_k,$$

where $P_1, \ldots, P_k$ are polynomials such that $\deg P_j \leq ej$. This fact is immediate by induction. Therefore,

$$\frac{(P^d)^{(k)}}{P^d} = \frac{P_1}{P} + \cdots + \frac{P_k}{P^k} = \frac{Q}{P^k},$$

where $\deg Q \leq ke$. Using this, one can check that each term in the determinant L_i can be written in the form

$$\frac{R}{f_0^{n-1} \cdots f_n^{n-1}},$$

where R is a polynomial of degree at most

$$(n-1)(n+1)e = (n^2 - 1)e.$$

Furthermore, one sees that since the denominator above is common for all the L_i, the two projective maps

$$(f_0^d, \ldots, f_n^d) \qquad \text{and} \qquad (R_0, \ldots, R_n)$$

are equal and that each R_i is a polynomial of degree at most $(n^2 - 1)e$. By comparing degrees and using the fact that $f_0, \ldots f_n$ are relatively prime, one gets

$$de \leq (n^2 - 1)e.$$

Therefore since f is not constant, $d \leq n^2 - 1$, and the proof of the proposition is complete.

I will now use the proof given in **[L1]** to obtain a p-adic analogue to a theorem of Green about analytic maps into Fermat hypersurfaces.

Theorem 3.2 (Green). *Let $f = (f_0, \ldots, f_n)\colon \mathbf{C}_p^\times \to \mathbf{P}^n$ be an analytic map into the Fermat variety X of degree d, so*

$$f_0^d + \cdots + f_n^d = 0.$$

Assume that $f_0, \ldots, f_n$ are analytic without common zeros. Suppose further that none of the functions are identically zero. Define an equivalence relation $i \sim j$ if f_i/f_j is constant. If $d \geq n^2$, then for each equivalence class S, one has

$$\sum_{i \in S} f_i^d = 0.$$

Proof: It suffices to prove the theorem when f is non-constant, and by induction, it suffices to show that if $n \geq 2$, then there is a proper subset I of $\{0, \ldots, n\}$ such that the functions $\{f_i^d\}$ for $i \in I$ are linearly dependent.

As in the previous proposition, let $g_i = f_i/f_0$ and $g = (1, g_1, \ldots, g_n)$. Then,

$$g_1^d + \cdots + g_n^d = -1.$$

If $n \geq 2$ and $g_1^d, \ldots, g_n^d$ are not linearly dependent, which we will assume in order to arrive at a contradiction, then by Cramer's Rule as before,

$$g_i^d = -\frac{L_i(g_1^d, \ldots, g_n^d)}{L(g_1^d, \ldots, g_n^d)} = -\frac{L_i}{L_0}.$$

Therefore, $f^d = (f_0^d, \ldots, f_n^d), g^d = (1, g_1^d, \ldots, g_n^d)$, and $\Lambda = (L_0, \ldots, L_n)$ all represent the same map into projective space. From the definition of height, one easily sees that

$$T_{g^d} = d\,T_g.$$

Hence,

$$\begin{aligned} d\,T_g(\rho) = T_L(\rho) &= \max_i v(1/L_i, \rho) + \max_i v(1/L_i, -\rho) - 2\max_i v(1/L_i, 0) \\ &\quad + \sum_{a \in C(-\rho,\rho)} \max_i \{\mathrm{ord}_a(1/L_i)\}(\rho - |v(a)|). \end{aligned}$$

Now,

$$\max_i v(1/L_i, \mu) \leq \max\{0, v(1/L_i, \mu)\} \leq \sum_i v^+(1/L_i, \mu).$$

Hence,

$$\max_i v(1/L_i, \rho) + \max_i v(1/L_i, -\rho) \leq \sum_i m_{L_i}(\rho).$$

But, L_i is the sum of products of logarithmic derivatives, so by Lemma II.3.1 and Proposition II.2.3,

$$\sum_i m_{L_i}(\rho) = O(\rho).$$

Since the f_i are analytic functions, the functions L_i can only have poles at the zeros of the functions f_i. Furthermore, since the f_i do not have common zeros, and since each term in the determinant expansion for L_i is a product of $n-1$

logarithmic derivatives, a pole of L_i can have at most order $n-1$. Therefore,

$$\begin{aligned}\sum_{a\in C(-\rho,\rho)} \max_i\{\mathrm{ord}_a(1/L_i)\}(\rho - |v(a)|) \\ &\le \sum_{a\in C(-\rho,\rho)} \max_i\{0, (\mathrm{ord}_a 1/L_i)\}(\rho - |v(a)|) \\ &\le (n-1)\sum_{i=0}^{n} N_{f_i,0}(\rho) \\ &\le (n-1)(n+1)T_g(\rho) + O(1).\end{aligned}$$

Combining these results, one finds

$$dT_g(\rho) \le (n^2-1)T_g(\rho) + O(\rho).$$

This implies that if $d \ge n^2$, then $T_g(\rho) = O(\rho)$, and hence each g_i is a rational function. Therefore, f is a polynomial map from $\mathbf{C}_p^\times$ into X and therefore extends to a rational map from $\mathbf{P}^1$ into X such that no proper subset of the functions f_i are linearly dependent. The proof is completed by using proposition 3.1 to conclude that $d < n^2$, and this contradiction completes the proof of the theorem.

Finally, the exact same proof as given in [**L1**], together with Theorem V.4.1, shows that the following perturbation of the Fermat hypersurface in $\mathbf{P}^3$ is p-adic Brody hyperbolic. This result is important because it shows that Brody hyperbolicity is not a "closed condition." Again, I include the proof here for the convenience of the reader.

Theorem 3.3 (Brody-Green). *Let d be an even integer ≥ 50. For $t \in \mathbf{C}_p$, let X_t be the variety in $\mathbf{P}^3$ defined by*

$$x_0^d + x_1^d + x_2^d + x_3^d + (tx_0x_1)^{d/2} + (tx_0x_2)^{d/2} = 0.$$

Then, for all but a finite number of $t \ne 0$, the variety X_t is p-adic Brody hyperbolic.

Proof: Let $f\colon \mathbf{C}_p^\times \to X_t$ be an analytic map, represented by analytic coordinate functions $(f_0, \ldots, f_3)$. To prove the theorem, one must show that f is constant.

The proof breaks up into several cases.

Case 1. Assume that some $f_i = 0$. Then, f is a map into the curve obtained by setting x_i equal to zero in the equation defining X_t. For all but a finite number of t, this curve will be non-singular of genus ≥ 2, and so by Theorem V.4.1 the map f is constant.

Case 2. Assume that $f_i \neq 0$ for all i. Let

$$\begin{aligned} g_0 &= f_0^2, & g_3 &= f_3^2, \\ g_1 &= f_1^2, & g_4 &= tf_0 f_1, \\ g_2 &= f_2^2, & g_5 &= tf_0 f_2. \end{aligned}$$

Then,

$$g = (g_0, \ldots, g_5) \colon \mathbf{C}_p^\times \to \mathbf{P}^5$$

is analytic and its image is contained in the Fermat hypersurface of degree $d/2$ in $\mathbf{P}^5$. Since $d/2 \geq 25 = 5^2$, one can use Theorem 3.2 to conclude that there is a partition of the set of indices $i = 0, \ldots, 5$ such that for each equivalence class S, one has

$$\sum_{i \in S} g_i^{d/2} = 0.$$

By the assumption that none of the f_i are zero, each equivalence class must have at least two elements.

Case 2(a). Assume that f_1/f_0 and f_2/f_0 are both constant. Then, the defining equation for X_t implies that

$$af_0^d + f_3^d = 0$$

for some constant a. Therefore, f_3/f_0 is constant, and hence f is constant.

Case 2(b). Assume that f_2/f_0 is constant but that f_1/f_0 is not constant. This implies that none of the ratios among g_0, g_1 and g_4 can be constant, and hence 0, 1 and 4 all belong to distinct equivalence classes. However, g_2/g_5 is constant. Thus, 0, 2 and 5 belong to the same equivalence class. This contradicts that fact that each equivalence class must have at least two elements, and hence this case does not occur.

Case 2(c). Assume that f_1/f_0 is constant but that f_2/f_0 is not constant. This case cannot occur for the same reason as case 2(b) by symmetry.

Case 2(d). Assume that both f_1/f_0 and f_2/f_0 are not constant. Then, as above, 0, 1 and 4 are in distinct equivalence classes. Furthermore, neither 5 nor 2 can be in the same equivalence class as 0. Therefore, $3 \sim 0$. One is left with two possibilities:

$$(*) \qquad 0 \sim 3, \quad 1 \sim 2, \quad 4 \sim 5,$$

or

$$(**) \qquad 0 \sim 3, \quad 1 \sim 5, \quad 2 \sim 4.$$

In case $(**)$, one has that $f_0 f_1/f_2^2$ and $f_0 f_2/f_1^2$ are constant. Therefore, f_2^3/f_1^3 is constant, and thus f_2/f_1 is constant. This means that $1 \sim 2$, which is a contradiction. That leaves case $(*)$. In this case there exist constants a and b such that

$$f_3 = af_0 \qquad \text{and} \qquad f_2 = bf_1.$$

The equation defining X_t then implies that

$$(1+a^d)f_0^d + (1+b^d)f_1^d + t^{d/2}(1+b^{d/2})(f_0 f_1)^{d/2} = 0.$$

The coefficients $(1+b^d)$ and $(1+b^{d/2})$ cannot both vanish, and this implies that f_1/f_0 is constant. This contradiction completes the proof of the theorem.

4. Nevanlinna Functions on Projective Varieties. I begin this section by defining the Nevanlinna functions for an analytic map $f\colon \mathbf{C}_p^\times \to \mathbf{P}^n$. After this, I will show how one can associate a height function to a divisor class on an arbitrary projective variety. I do this to illustrate some of the functorial properties of height functions and because one of these properties will be useful in the proof of Cartan's Theorem in the next section.

Let $x_0, \ldots, x_n$ be projective coordinates for $\mathbf{P}^n$, and let $(f_0, \ldots, f_n)$ be analytic functions without common zeros representing a map $f\colon \mathbf{C}_p^\times \to \mathbf{P}^n$. Let H_j be the hyperplane defined by $x_j = 0$. If the image of f is not contained in H_j (i.e. $f_j \not\equiv 0$), then define the **counting function** with respect to H_j by

$$N_{f,H_j}(\rho) = \sum_{a \in C(-\rho,\rho)\cdot} \max_i \{\mathrm{ord}_a(f_j/f_i)\}(\rho - |v(a)|),$$

and the **mean proximity function** with respect to H_j by

$$m_{f,H_j}(\rho) = \max_i\{v^+(f_j/f_i, \rho)\} + \max_i\{v^+(f_j/f_i, -\rho)\}.$$

Then, as usual, define the **height** with respect to H_j by

$$T_{f,H_j} \doteq m_{f,H_j} + N_{f,H_j}.$$

Now, for an arbitrary hyperplane H, let $\psi\colon \mathbf{P}^n \to \mathbf{P}^n$ be an invertible projective linear transformation taking H to the hyperplane defined by $x_0 = 0$. Then, define

$$N_{f,H} = N_{\psi\circ f, H_0}, \quad m_{f,H} = m_{\psi\circ f, H_0} \quad \text{and} \quad T_{f,h} = T_{\psi\circ f, H_0},$$

provided that the image of f is not contained in H so that $\psi \circ f$ is non-constant. The following proposition tells us that these functions are well-defined up to a bounded term.

Proposition 4.1. *The counting function $N_{f,H}$ is well-defined. The height function $T_{f,H}$ and mean proximity function $m_{f,H}$ are well-defined up to a bounded term. In fact,*

$$T_{f,H}(\rho) = T_f(\rho) + O(1),$$

where T_f is the height of the map f into $\mathbf{P}^n$ defined in Section 2, and the $O(1)$ term depends only on H.

Proof: Clearly the counting function does not depend on the choice of linear mapping ψ. The equality between $T_{f,H}$ and T_f follows directly from the definitions and Proposition 2.1. Hence $m_{f,H}$ is also well-defined modulo $O(1)$.

Remark. Note that if the image of f is contained in a hyperplane H, we cannot define the mean-proximity or counting functions. However, the above proposition tells us we can always define the height function $T_{f,H}$ to be T_f.

The next proposition says that if the image of a map f is contained in a proper linear subspace $\mathbf{P}^n$ of $\mathbf{P}^N$, then as long as one is interested only modulo bounded terms, the Nevanlinna functions of f considered as a map into $\mathbf{P}^n$ are the same as those if f is considered as a map into $\mathbf{P}^N$.

Proposition 4.2. *Let $f\colon \mathbf{C}_p^\times \to \mathbf{P}^n \subsetneq \mathbf{P}^N, 1 \le n < N$ be an analytic map contained in a proper linear subspace of $\mathbf{P}^N$. Let $T_{f,\mathbf{P}^N}$ and $T_{f,\mathbf{P}^n}$ denote the height of the map f considered as a map into $\mathbf{P}^N$ and $\mathbf{P}^n$ respectively. Then*

$$T_{f,\mathbf{P}^N}(\rho) = T_{f,\mathbf{P}^n}(\rho) + O(1),$$

where the $O(1)$ term depends only on the embedding $\mathbf{P}^n \hookrightarrow \mathbf{P}^N$. Furthermore, if H is a hyperplane in $\mathbf{P}^N$ such that $H' = H \cap \mathbf{P}^n$ is a hyperplane in $\mathbf{P}^n$ (i.e. $\mathbf{P}^n \not\subset H$) and the image of f is not contained in H, then

$$N_{f,H',\mathbf{P}^n}(\rho) = N_{f,H,\mathbf{P}^N}(\rho)$$

and

$$m_{f,H',\mathbf{P}^n}(\rho) = m_{f,H,\mathbf{P}^N}(\rho) + O(1),$$

where the $O(1)$ term depends only on H and the embedding $\mathbf{P}^n \hookrightarrow \mathbf{P}^N$.

Proof: The statement about the counting function is clear, so the statement about the mean proximity function follows immediately from the statement about the heights and Proposition 4.1. To prove the statement about the heights, choose projective coordinates $x_0, \ldots, x_N$ in $\mathbf{P}^N$ so that

$$\mathbf{P}^n = \{(x_0, \ldots, x_N) : x_j = 0 \text{ for } n+1 \le j \le N\}.$$

Then the statement is obvious from the definitions and the choice of coordinates is what introduces the $O(1)$ term in the equality.

For the rest of this section, I follow Chapter 4 of [**L3**] in order to define height functions on arbitrary projective varieties. Let $X \subset \mathbf{P}^N$ be a projective algebraic variety. Then, in this chapter, by an analytic map $f: \mathbf{C}_p^\times \to X$, I mean an analytic map $f: \mathbf{C}_p^\times \to \mathbf{P}^N$ such that the image of f is contained in X.

Proposition 4.3. *Let X be a projective algebraic variety. Let*

$$\phi: X \to \mathbf{P}^n \quad \textit{and} \quad \psi: X \to \mathbf{P}^m$$

be algebraic maps from X into projective spaces. Define the map

$$\phi \otimes \psi: X \to \mathbf{P}^{(n+1)(m+1)-1}$$

by

$$x \mapsto (\ldots, \phi_i(x)\psi_j(x), \ldots).$$

Let $f: \mathbf{C}_p^\times \to X$ be an analytic map. Then

$$T_{\phi\otimes\psi\circ f} = T_{\phi\circ f} + T_{\psi\circ f}.$$

Proof: This is obvious since

$$\max_{i,j} v((1/\phi_i \circ f)(1/\psi_i \circ f), \mu) = \max_i v(1/\phi_i \circ f, \mu) + \max_j v(1/\psi_j \circ f, \mu).$$

Let X be a projective algebraic variety over $\mathbf{C}_p$. Consider pairs (U, ϕ) consisting of a Zariski open set U and a non-zero rational function ϕ on U. Two such pairs, (U, ϕ) and (V, ψ) are equivalent if ϕ/ψ is a non-zero constant on $U \cap V$. A maximal family of equivalent pairs $\{(U_i, \phi_i)\}$ such that the open sets U_i cover X is called a (Cartier) **divisor** D. A family of pairs $\{(U_i, \phi_i)\}$ such that the U_i cover X but which is not maximal is called a representation of D. Note that D can be identified with a codimension one subset of X with multiplicities by taking the zeros and poles of the functions ϕ_i. If a divisor D can be represented by $\{(U_i, \phi_i)\}$, where all the ϕ_i are polynomials (i.e. without poles), then D is called **effective.** If there exists a rational function ϕ defined on all of X such that (X, ϕ) represents the divisor D, then D is called **principal**, or **linearly equivalent to 0.**

Given two divisors, their sum $\{(U_i, \phi_i)\} + \{(V_j, \psi_j)\}$ is defined to be the divisor represented by $\{(U_i \cap V_j, \phi_i\psi_j)\}$. Two divisors, D and D' are called **linearly equivalent** if $D - D'$ is a principal divisor. Linear equivalence classes of divisors form an Abelian group with the divisor class represented by $(X, 1)$ as the zero element.

To a divisor D on X, one associates the **line sheaf** (i.e. the sheaf of sections of a line bundle) $\mathcal{L}_D$ such that if (U, ϕ) represents D on U, then

$$\mathcal{L}_D(U) = \mathcal{O}_X(U)\phi^{-1},$$

which is the set of all rational functions ψ on U such that $\psi\phi$ does not have any poles in U. A **global section** $s \in H^0(X, \mathcal{L}_D)$ is then a rational function on X such that if $\{(U_i, \phi_i)\}$ represents D, then for all i, $s\phi_i$ does not have any poles in U_i. Given a global section $s \in H^0(X, \mathcal{L}_D)$, the set of pairs $\{(U_i, s\phi_i)\}$ represents an effective divisor linearly equivalent to D. This divisor is denoted by $(s)_0$ and is called the **divisor of zeros** of s.

The set of global sections $H^0(X, \mathcal{L}_D)$ is a finite dimensional $\mathbf{C}_p$ vector space. Given a section $s \in H^0(X, \mathcal{L}_D)$ and a point $x \in X$, denote by $s(x)$ any one of the values $(s\phi)(x)$, where (U, ϕ) represents D in a neighborhood of x. By a **linear system,** I mean a non-trivial vector subspace $V \subseteq H^0(X, \mathcal{L}_D)$. Note that this is a slightly non-standard definition of the term. By a divisor D in V, I mean a divisor D_0 such that $D_0 = (s)_0$ for some s in V. A point x in X is called a **base point** for the linear system V if $s(x) = 0$ for every s in V. A linear system is said to be **base point free** if for every $x \in X$, there exists a section $s \in V$ such that $s(x) \neq 0$.

If $S = \{s_0, \ldots, s_n\}$ span a linear system V and V is base point free, then the map

$$\Phi(x) = (s_0(x), \ldots, s_n(x))$$

is a morphism from X into $\mathbf{P}^n$. Note that although the sections s_j are not well-defined functions on X, they still determine a well-defined map into $\mathbf{P}^n$ because they are not defined only up to the transition functions of $\mathcal{L}_D$, which are non-zero constants. We can use this morphism to define the height of a function $f\colon \mathbf{C}_p^\times \to X$ relative to the linear system V and the choice of spanning sections S by defining

$$T_{f,V,S}(\rho) = T_{\Phi \circ f}(\rho).$$

We will now see that, modulo a bounded term, this height function depends only on the divisor D and not on V or S. We will need the following lemma.

Lemma 4.4. *Let X be a projective algebraic variety over $\mathbf{C}_p$. Let $\phi_1, \ldots, \phi_n$ be rational functions on X whose divisors of zeros have no points in common. Then, there exists a constant C such that for any analytic map $f\colon \mathbf{C}_p^\times \to X$ such that $\phi \circ f \not\equiv \infty$,*

$$\max_i \{v(1/\phi_i \circ f, \mu)\} \geq C,$$

for all μ.

Proof: By assumption, $(\phi_1, \ldots, \phi_n)$ is the unit ideal in $\mathbf{C}_p[\phi_1, \ldots, \phi_n]$, for otherwise $\phi_1, \ldots, \phi_n$ would have a common zero in X. Therefore,

$$1 = \sum a_{\nu_1 \cdots \nu_n} \phi_1^{\nu_1} \cdots \phi_n^{\nu_n}.$$

Hence,

$$\begin{aligned}0 &= -v(1 \circ f, \mu)\\ &= -v\left(\sum a_\nu(\phi_1 \circ f)^{\nu_1} \cdots (\phi_n \circ f)^{\nu_n}, \mu\right)\\ &\le -\sum_\nu v(a_\nu) - \sum_\nu (\nu_1 v(\phi_1 \circ f, \mu) + \cdots + \nu_n v(\phi_n \circ f, \mu)).\end{aligned}$$

Therefore,

$$\sum_\nu v(a_\nu) \le \sum_\nu \nu_1 v(1/\phi_1 \circ f, \mu) + \cdots + \nu_n v(1/\phi_n \circ f, \mu)$$

for all f and all μ, and hence the lemma.

Theorem 4.5. *Let X be a projective algebraic variety. Let D be an effective divisor on X. Let $V, V' \subseteq H^0(X, \mathfrak{L}_D)$ be base point free linear systems spanned by $S = \{s_0, \ldots, s_m\}$ and $S' = \{s'_0, \ldots, s'_n\}$ respectively . Then*

$$T_{f,V,S}(\rho) = T_{f,V',S'}(\rho) + O(1).$$

Proof: Let E_0 be the effective divisor given as the divisor of zeros of s_0. Let $E_i, i = 1, \ldots m$ be the effective divisors such that if each section s_i is considered as a rational function on X, rather than a global section of $\mathfrak{L}_D$, then

$$(s_i) = E_i - E_0.$$

Similarly, let E'_j be the effective divisors such that

$$(s'_j) = E'_j - E_0$$

when s'_j is considered as a rational function, rather than a section of $\mathfrak{L}_D$.

Now, consider the rational functions

$$s_0/s'_j, \ldots, s_m/s'_j$$

for some j. Because $(s_i/s'_j) = E_i - E'_j$ and since E'_j is effective and V is base point free, by Lemma 4.4 there exists a constant C_j such that for any analytic map $f\colon \mathbf{C}_p^\times \to X$ whose image is not contained in E'_j, one has

$$\max_i\{v(1/(s_i/s'_j) \circ f, \mu)\} \ge C_j.$$

Therefore, by applying the lemma for each j, there exists a constant C' such that for every analytic map $f\colon \mathbf{C}_p^\times \to X$ whose image is not contained in D, one has

$$\max_i\{v(1/s_i \circ f, \mu)\} \geq C' \max_j\{v(1/s'_j \circ f, \mu)\}$$

for all μ. The constant C' depends on the divisor E_0. However, since V is base point free,

$$X = \bigcup_i (X - \mathrm{Supp}\ E_i).$$

Therefore, there exists a constant C so that

$$\max_i\{v(1/s_i \circ f, \mu)\} \geq C \max_j\{v(1/s'_j \circ f, \mu)\}$$

for every μ and for every analytic map $f\colon \mathbf{C}_p^\times \to X$. Hence by symmetry,

$$T_{f,V,S} = T_{f,V',S'} + O(1).$$

Let D be a divisor on X. Assume that there exists a base point free linear system $V \subseteq H^0(X, \mathcal{L}_D)$, and let $S = \{s_0, \ldots, s_n\}$ span V. Given any analytic map $f\colon \mathbf{C}_p^\times \to X$, define

$$T_{f,D}(\rho) = T_{f,V,S}(\rho).$$

Then by the above theorem, $T_{f,D}$ is well-defined modulo a bounded term. Furthermore, if D and D' are divisors, and

$$V \subseteq H^0(X, \mathcal{L}_D) \quad \text{and} \quad V' \subseteq H^0(X, \mathcal{L}_{D'})$$

are base point free linear systems spanned by

$$S = \{s_0, \ldots, s_m\} \quad \text{and} \quad S' = \{s'_0, \ldots s'_n\} \quad \text{respectively,}$$

then

$$\{\ldots, s_i s'_j, \ldots\},\ i = 0, \ldots m,\ j = 0 \ldots n$$

spans a base point free linear system for $D + D'$. Therefore, Proposition 4.3 implies

Proposition 4.6

$$T_{f,D}(\rho) + T_{f,D'}(\rho) = T_{f,D+D'}(\rho) + O(1).$$

In order to associate a height function to any divisor D, one needs to recall the notion of a very ample divisor. A divisor E is called **very ample** if the complete linear system associated to E is base point free and the associated morphism to projective space is a closed embedding. (By complete linear system, I mean the entire vector space $H^0(X, \mathcal{L}_E)$.) If E is very ample and effective, then E is the pull-back under the embedding given by E of a hyperplane in projective space. A fundamental result in algebraic geometry is that any divisor D can be written as the difference of two very ample divisors. Furthermore, the sum of two very ample divisors is again very ample

Given any divisor D on X, let E and F be very ample effective divisors such that $D = E - F$. Given an analytic map $f: \mathbf{C}_p^\times \to X$, define

$$T_{f,D}(\rho) = T_{f,E}(\rho) - T_{f,F}(\rho).$$

This is well-defined up to a bounded term because if $D = E' - F'$ then $E + F'$ is linearly equivalent to $E' + F$, so

$$T_{f,E+F'}(\rho) = T_{f,E'+F}(\rho) + O(1),$$

by Theorem 4.5, and therefore

$$T_{f,E}(\rho) - T_{f,F}(\rho) = T_{f,E'}(\rho) - T_{f,F'}(\rho) + O(1),$$

by the additivity of the height function for base point free linear systems (Proposition 4.6).

In this section we have shown how to associate a height function to an analytic map $f: \mathbf{C}_p^\times \to X$ and a divisor D on X. The following theorem (see [**L1**] pages 215-216) summarizes the important properties.

Theorem 4.7. *Let $f: \mathbf{C}_p^\times \to X$ be an analytic map into a projective variety X over $\mathbf{C}_p$.*

(1) If D is linearly equivalent to D' then

$$T_{f,D}(\rho) = T_{f,D'}(\rho) + O(1).$$

(2) If D and D' are divisors on X, then

$$T_{f,D+D'}(\rho) = T_{f,D}(\rho) + T_{f,D'}(\rho) + O(1).$$

(3) If E is a very ample divisor on X and $\psi: X \to \mathbf{P}^n$ is an associated projective embedding, then

$$T_{f,E}(\rho) = T_{\psi \circ f}(\rho) + O(1),$$

where $T_{\psi \circ f}$ is the Cartan-Nevanlinna height defined in Section 2.

(4) If D is any divisor and E is a very ample divisor then

$$T_{f,D} = O(T_{f,E}).$$

(5) If D is an effective divisor and the image of f is not contained in D, then

$$T_{f,D} \geq -O(1).$$

(6) The association $(f, D) \mapsto T_{f,D}$ is functorial in (X, D), meaning if $\psi: X \to Y$ is an algebraic morphism and $D = \psi^* D'$, where D' is a divisor on Y, then

$$T_{f,D}(\rho) = T_{\psi \circ f, D'}(\rho) + O(1).$$

(7) Let $\psi: X \to \mathbf{P}^N$ and let H be a hyperplane in $\mathbf{P}^N$. Let $D = \psi^* H$. Then,

$$T_{f,D}(\rho) = T_{\psi \circ f}(\rho) + O(1).$$

Proof: Statement (1) is essentially Theorem 4.5, (2) is Theorem 4.5 together with Proposition 4.6, and (3) follows directly from the definitions and Theorem 4.5.

To show (4), note that there is an integer n such that $nE - D$ is very ample. Therefore,

$$nT_{f,E}(\rho) - T_{f,D}(\rho) \geq -O(1)$$

by (3), so

$$T_{f,D}(\rho) \leq nT_{f,E}(\rho) + O(1).$$

Similarly,

$$-T_{f,D}(\rho) \leq mT_{f,E}(\rho) + O(1),$$

so $T_{f,D} = O(T_{f,E})$.

To show (5), let $D = E - F$, where E and F are very ample and effective, so $D + F = E$. Since D is also effective,

$$H^0(X, \mathcal{L}_F) \subset H^0(X, \mathcal{L}_E),$$

so any base point free linear system associated to F can be extended to a base point free linear system associated to E. Therefore, by the definition of the height,

$$T_{f,F} \leq T_{f,E} + O(1),$$

which is what needed to be shown.

Statement (6) follows directly from the definitions, and (7) follows from (6) and (3).

Remark. In particular, if D is linearly equivalent to zero, then part (1) above says that $T_{f,D}(\rho) = O(1)$, and this is sometimes known as the **First Main Theorem.**

5. Cartan's Second Main Theorem for $\mathbf{P}^n$. This section contains a p-adic analogue of Cartan's theorem for holomorphic curves in $\mathbf{P}^n$, as improved by Lang in [**L1**]. This theorem is the analogue for analytic maps from $\mathbf{C}_p^\times$ into $\mathbf{P}^n$ of Nevanlinna's Second Main Theorem. This theorem is sometimes called, "Cartan's Second Main Theorem for $\mathbf{P}^n$," or "Cartan's Hyperplane Theorem."

Theorem 5.1. *Let $f = (f_0, \ldots, f_n)\colon \mathbf{C}_p^\times \to \mathbf{P}^n$ be an analytic map. Choose the coordinate functions $f_0, \ldots, f_n$ representing f to be analytic functions on $\mathbf{C}_p^\times$ without common zeros. Assume that the image of f is not contained in any hyperplane. Let $H_1, \ldots, H_q$ be q hyperplanes in general position. Let*

$$W = W(f) = W(f_0, \ldots, f_n)$$

be the Wronskian of f. Then,

$$\sum_{k=1}^{q} m_{f,H_k}(\rho) + T_{f,K}(\rho) + N_{1/W}(\rho) \leq O(\rho),$$

where the **canonical height** *$T_{f,K}$ on $\mathbf{P}^n$ is defined modulo $O(1)$ by*

$$T_{f,K} = -(n+1)T_{f,H},$$

for any hyperplane H.

Proof: Since any collection of less than $n+2$ linearly independent hyperplanes can be extended to a set of $n+2$ hyperplanes in general position and $m_{f,H}(\rho) \geq -O(1)$, it suffices to consider $q \geq n+2$. Let $r = q - (n+1)$, and let $M = \{m_1, \ldots, m_p\}$ be a subset of $\{1, \ldots, q\}$ with cardinality r. Also, let

$$H_M(x) = H_{m_1}(x) \cdots H_{m_r}(x)$$

for x in $\mathbf{P}^n$, where by abuse of notation we have also used $H_j(x)$ to denote the linear form defining the hyperplane H_j.

First, we will make some observations about hyperplanes. By the definition of general position, given x in $\mathbf{P}^n$, there exists a set M as above such that $H_M(x) \neq 0$. This gives a map $\psi\colon \mathbf{P}^n \to \mathbf{P}^N$ given by

$$x \mapsto (H_{M_0}(x), \ldots, H_{M_N}(x))$$

where $M_0, \ldots, M_N$ are all the subsets of $\{1, \ldots, q\}$ with cardinality r.

Let H be any hyperplane. We have already remarked that

$$T_{f,H} = T_{f,H_j} + O(1) \quad \text{and so} \quad r\,T_{f,H} = T_{f,H_{m_1}} + \cdots + T_{f,H_{m_r}} + O(1).$$

Furthermore, since $H_{m_1} + \cdots + H_{m_r}$ is the inverse image under ψ of a hyperplane in $\mathbf{P}^N$, Theorem 4.7 (7) implies

$$r\,T_{f,H} = T_{\psi \circ f} + O(1),$$

where $T_{\psi \circ f}$ is the height of $\psi \circ f$ as a map into $\mathbf{P}^N$, which by definition (see Section 2) is given by

$$T_{\psi \circ f}(\rho) = \max_{|M|=r} v\left(\frac{1}{H_M \circ f}, \rho\right) + \max_{|M|=r} v\left(\frac{1}{H_M \circ f}, -\rho\right) + O(1).$$

Now, the idea is to rewrite things so that we can apply the Lemma on the Logarithmic Derivative. Given an $(n+1)$-tuple

$$I = \{i_1, \ldots, i_{n+1}\} \subset \{1, \ldots, q\},$$

let d_I be the determinant of the linear transformation taking the standard projective coordinates on $\mathbf{P}^n$ to $H_{i_1}, \ldots, H_{i_{n+1}}$. Let $h_i = H_i \circ f$. Then,

$$d_I W(h_{i_1}, \ldots, h_{i_{n+1}}) = W(f_0, \ldots, f_n).$$

Let

$$L_I = L(h_{i_1}, \ldots, h_{i_{n+1}}) = \frac{W(h_{i_1}, \ldots, h_{i_{n+1}})}{h_{i_1} \cdots h_{i_{n+1}}}.$$

Finally, let

$$G = \frac{h_1 \cdots h_q}{W(f_0, \ldots, f_n)}.$$

Then, whenever $\{1, \ldots, q\} = I \cup M$ is decomposed into complementary sets, one has

$$H_M \circ f = h_{m_1} \cdots h_{m_p} = G d_I L_I.$$

We will see that G has small height, and we will be able to apply the Lemma on the Logarithmic Derivative to L_I as follows.

One has

$$\begin{aligned}
\max_I v(1/L_I, \rho) &+ \max_I v(1/L_I, -\rho) \\
&\leq \sum_I v^+(1/L_I, \rho) + \sum_I v^+(1/L_I, -\rho) \\
&= \sum_I m_{L_I}(\rho) \\
&\leq O(\rho),
\end{aligned}$$

where the last inequality follows from Lemma II.3.1 by noting that L_I is the sum of products of logarithmic derivatives.

The estimate for G goes as follows:

$$\begin{aligned} v(1/G,\rho)+v(1/G,-\rho) =&\ v^+(1/G,\rho)+v^+(1/G,-\rho) \\ &\qquad -v^+(G,\rho)-v^+(G,-\rho) \\ =&\ m_G(\rho)-m_{1/G}(\rho) \\ \text{[Jensen's Formula]}\quad =&\ N_{1/G}(\rho)-N_G(\rho)+O(1) \\ =&\sum_{j=1}^{q} N_{1/h_j}(\rho)-\sum_{j=1}^{q} N_{h_j}(\rho) \\ &\qquad -N_{1/W}(\rho)+N_W(\rho)+O(1), \end{aligned}$$

where the last equality follows from the definition of G. Note that

$$N_{h_j}=N_W=0$$

for all j since these functions are analytic. Therefore, one gets

$$\begin{aligned} r\,T_{H\circ f} &= \max_M v\left(\frac{1}{H_M\circ f},\rho\right)+\max_M v\left(\frac{1}{H_M\circ f},-\rho\right)+O(1) \\ &= \max_I v\left(\frac{1}{Gd_IL_I},\rho\right)+\max_I v\left(\frac{1}{Gd_IL_I},-\rho\right)+O(1) \\ &\leq \sum_{j=1}^{q} N_{f,H_j}(\rho)-N_{1/W}(\rho)+O(\rho). \end{aligned}$$

This implies

$$\sum_{j=1}^{q} m_{f,H_j}(\rho)+(r-q)T_{f,H}(\rho)+N_{1/W}(\rho)\leq O(\rho),$$

by using Jensen's formula and the fact that

$$T_{f,H}=T_{f,H_j}+O(1).$$

The proof is completed by recalling that $r-q=-(n+1)$.

Paul Vojta [**Vo**] has a generalization of Cartan's Theorem over the complex numbers, whose statement is as follows:

Theorem. *Let $H_1,\ldots,H_{n+2}$ be hyperplanes in general position in $\mathbf{P}^n$. Then, there exists a finite set $\mathcal{R}$ of proper linear subspaces of $\mathbf{P}^n$ such that for every* *holomorphic map* $f: \mathbf{C} \to \mathbf{P}^n$ *whose image does not lie in some element of* $\mathcal{R}$, *for every* $\varepsilon > 0$, *and for all* r *outside a set (depending on* f*) of finite Lebesgue measure,*

$$\sum_{i=1}^{n+2} m_{f,H_i}(r) \leq (n+1+\varepsilon)T_f(r) + O(1).$$

The point of this generalization is that the functions which do not satisfy the above inequality must not only be degenerate, but must be degenerate in the very precise sense that their images must be contained in a specific *finite* set of linear subspaces, which depend only on the hyperplanes $H_1, \ldots, H_{n+2}$. Vojta's result is technically in the category of results which depend only on functorial properties of heights, the Lemma on the Logarithmic Derivative, and projective linear algebra. However, the linear algebra involved is extremely technical, and here functoriality of heights includes Silverman's notion of arithmetic distance functions [**Si**]. Therefore, I have decided that a more detailed description of this result would take us further afield than is worth going at this point.

CHAPTER IV

Berkovich Theory

In this chapter, the analytic theory developed by Vladimir Berkovich in [**Ber**] will be outlined. This theory allows him to use ordinary topological and geometric techniques to study p-adic analytic maps into smooth projective algebraic curves. When reading [**Ber**], one should also be familiar with the fundamentals of algebraic geometry and with rigid analytic geometry as in [**BGR**], [**BL1**] and [**BL2**]. Thus, this chapter requires considerably more background on the part of the reader than the previous chapters did.

The canonical topology on $\mathbf{C}_p^\times$, which is induced by the p-adic norm, is totally disconnected. This means that one cannot use many of the topological techniques available in complex analysis. For example, the topological theory of covering spaces and map liftings does not work on totally disconnected spaces. Analytic continuation is another problem on such spaces. If an analytic function were defined to be a function which locally has a power series representation, then the function which is zero on the "closed" unit disc and one outside the "closed" unit disc would qualify as an analytic function. This means that there could be no reasonable theory of analytic continuation. There are a number of reasonable definitions for analytic function which have analytic continuation properties. One way to go about this is to define "special" open sets and to require an analytic function to satisfy some property on a special open set in the neighborhood of each point. This effectively "connects" the space by reducing the number of open sets. The most general approach is that of Tate, whose work shows, as Grothendieck has pointed out, that to do geometry, one does not need a reasonable space, only a cohomology theory.

However, Berkovich [**Ber**] has found a nice topological space which corresponds to this cohomology theory. Given a rigid analytic space, Berkovich uses a concept from spectral theory in order to densely embed the rigid analytic space into a space with nice topological properties, such as arc-connectedness and local contractibility, so that ordinary topological techniques can be used in the study of rigid analytic geometry. Berkovich is then able to use the topological concepts of universal coverings and map liftings to prove that there is no non-constant analytic map from $\mathbf{C}_p$ to a smooth projective curve of genus ≥ 1.

This chapter outlines the foundations of Berkovich's theory and is meant as an overview. Some technicalities will be ignored, and many proofs will be omitted in order not to interfere with the basic ideas and examples. Furthermore, even though the results in this section hold over more general classes of complete non-Archimedean fields, I will continue to work only with $\mathbf{C}_p$ because I do not want to be careful about specifying hypotheses on the ground field. The interested reader can look at [**Ber**] where everything is carefully worked out in general.

1. The Closed Ball. Let A be a commutative Banach ring with identity. Eventually, A will be restricted to a certain class of $\mathbf{C}_p$-algebras, but for now A is arbitrary. The spectrum $\mathcal{M}(A)$ is the set of all bounded multiplicative semi-norms on A provided with the weakest topology with respect to which all real valued functions on $\mathcal{M}(A)$ of the form

$$|\ | \mapsto |f|, \quad f \in A$$

are continuous. This definition has some immediate, though not trivial, topological consequences.

Theorem 1.1 *([**Ber**] 1.2.1). The spectrum $\mathcal{M}(A)$ is a non-empty compact Hausdorff space.*

Remark. Berkovich proves this theorem first for the case when $A = \prod_{i \in I} K_i$ is a product of complete non-Archimedean fields. In this case, he is able to show that $\mathcal{M}(A)$ is homeomorphic to the Stone-Čech compactification of the discrete set I. The general case follows because any Banach ring A can be mapped to a product of fields in such a way that the induced map from $\mathcal{M}(\prod_{i \in I} K_i)$ to $\mathcal{M}(A)$ is surjective. Thus, $\mathcal{M}(A)$ is the continuous image of a compact set, and hence compact. However, the discrete set I may be rather large, so whereas $\mathcal{M}(A)$ is always compact, it is rarely sequentially compact because it will fail to satisfy the first axiom of countability. Thus, one thing that Berkovich's theory does not let us do is use compactness to take subsequences.

The most important Banach ring for our purposes is the **Tate algebra**

$$\mathbf{C}_p < z_1, \ldots, z_n >,$$

which is the set of power series

$$\sum \vec{a}_{\vec{k}}\, \vec{z}^{\vec{k}}$$

converging on the closed unit ball, or in other words, such that $|\vec{a}_{\vec{k}}|_p \to 0$ as $|\vec{k}| \to \infty$, where we have used multi-index notation. Similarly, let

$$\mathbf{C}_p < r_1^{-1} z_1, \ldots, r_n^{-1} z_n >$$

denote the set of power series

$$\sum \vec{a}_{\vec{k}}\, \vec{z}^{\vec{k}}$$

such that $|\vec{a}_{\vec{k}}|_p\, \vec{r}^{\vec{k}} \to 0$ as $|\vec{k}| \to \infty$.

The **closed n-ball**, in the sense of Berkovich, of radius $\vec{r}$, where $\vec{r}$ is an n-tuple of non-negative real numbers, is defined by

$$\mathbf{B}^n(\vec{0},\vec{r}) = \mathcal{M}(\mathbf{C}_p\!<\!r_1^{-1}z_1,\ldots,r_n^{-1}z_n\!>).$$

Note that the space $\mathbf{B}^n(\vec{0},\vec{r})$ is compact by the above theorem. The Tate algebra above will be denoted $T^n(\vec{r})$ for brevity. I will also use

$$T^n \text{ to denote } T^n(0,(1,\ldots,1)), \qquad \text{and } \mathbf{B}^n \text{ to denote } \mathbf{B}^n(0,(1,\ldots,1)).$$

We will now examine the space $\mathbf{B}^1(0,r)$ in some detail. Of course, the space $\mathbf{B}^1(0,r)$ should contain the elements a in $\mathbf{C}_p$ such that $|a|_p \le r$. Indeed, such an element determines a bounded multiplicative semi-norm $|\ |_a$ on $T^1(0,r)$ defined by

$$|f|_a = |f(a)|_p.$$

These points are called points of *type (1)*. Another way to get a point in $\mathbf{B}^1(0,r)$ is as follows. Let $E = E(a,\rho)$ denote the closed disc in $\mathbf{C}_p$ around the point a with radius ρ. If E is in $\mathbf{B}^1(0,r)$, then a function f in $T^1(r)$ has a power series expansion

$$f(z) = \sum_{k=0}^{\infty} c_k(z-a)^k$$

around the point a which has radius of convergence at least ρ. Then, the semi-norm $|\ |_E$ defined by

$$|f|_E = \sup |c_k|_p\, \rho^k$$

is bounded and multiplicative. Multiplicativity follows from the non-Archimedean property of $|\ |_p$. If $\rho \in |\mathbf{C}_p|_p$, then the resulting point in $\mathbf{B}^1(0,r)$ is called a point of *type (2)*. Otherwise, it is called a point of *type (3)*. There is one more way to get a point in $\mathbf{B}^1(0,r)$. Let $\mathcal{E}$ be a family of closed discs E. The family $\mathcal{E}$ is called an **embedded family of closed discs** if given any two discs E and E' in $\mathcal{E}$ such that the radius of E is less than or equal to the radius of E', then E is contained in E'. Define a bounded, multiplicative semi-norm $|\ |_{\mathcal{E}}$ by

$$|f|_{\mathcal{E}} = \inf_{E\in\mathcal{E}} |f|_E.$$

It is a fundamental fact that the intersection of all the closed discs in an embedded family of closed discs is either (a) a single point, (b) a closed disc, or (c) empty. In the first two cases, the semi-norm $|\ |_{\mathcal{E}}$ is the same as the semi-norm defined by the intersection as above. However, if the intersection is empty, then this semi-norm gives a new point, called a point of *type (4)*. It turns out that any point in $\mathbf{B}^1(0,r)$ is of one of these four types.

Proposition 1.2 *([Ber] 1.4.4). Let* $|\cdot|$ *be a bounded multiplicative semi-norm on* $T^1(r)$ *corresponding to a point of* $\mathbf{B}^1(0,r)$*. Then* $|\ |$ *is* $|\ |_{\mathcal{E}}$ *for some embedded family of discs* $\mathcal{E}$*. The point is of type (1), (2), (3), or (4), depending on whether the intersection of* $\mathcal{E}$ *is (1) a point, (2) a closed disc with radius an element of* $|\mathbf{C}_p^\times|_p$*, (3) a closed disc with radius not an element of* $|\mathbf{C}_p^\times|_p$*, or (4) empty.*

Proof: Multiplicative semi-norms on $T^1(r)$ are completely determined by their values on the functions $z-a$ for $|a|_p \leq r$. Consider the family of closed discs

$$\mathcal{E} = \{E(a, |z-a|) : |a|_p \leq r\}.$$

I claim that if $|z-a| \leq |z-b|$, then

$$E(a, |z-a|) \subset E(b, |z-b|).$$

To show this, it suffices to show that $|a-b|_p \leq |z-b|$, which is true since

$$\begin{aligned} |a-b|_p = |a-b| &= |(z-b)-(z-a)| \\ &\leq \max\{|z-a|, |z-b|\} = |z-b|, \end{aligned}$$

where in the expression $|a-b|$, we consider $a-b$ as a constant function in $T^1(r)$. Therefore, the family $\mathcal{E}$ is a family of embedded closed discs. By construction,

$$|z-a|_{\mathcal{E}} = \inf_{|b|_p \leq r} |z-a|_{E(b,|z-b|)} = \inf_{|b|_p \leq r} \max\{|z-b|, |a-b|_p\} = |z-a|,$$

where the last equality follows from the fact that

$$\text{if} \quad |a-b|_p < |z-a|, \quad \text{then} \quad |z-b| = |z-a+a-b| = |z-a|.$$

Therefore, $|\ | = |\ |_{\mathcal{E}}$, and the proof is complete.

To get a more intuitive feel for what is happening here, it is helpful to keep the following picture in mind. Fix a point a in $\mathbf{C}_p$ such that $|a|_p \leq r$. Then, the space $\mathbf{B}^1(0,r)$ contains the line segment consisting of the points corresponding to the semi-norms

$$|\ |_{E(a,\rho)}, \qquad \text{where} \qquad 0 \leq \rho \leq r.$$

Note that

$$|\ |_{E(a,0)} = |\ |_a \qquad \text{and} \qquad |\ |_{E(a,r)} = |\ |_{E(0,r)}.$$

More generally, note that if $\rho \geq |a-b|_p$, then

$$|\ |_{E(a,\rho)} = |\ |_{E(b,\rho)}.$$

Thus, Berkovich's theory connects the totally disconnected space by adding a distinguished point $|\ |_{E(0,r)}$ and then adding a line segment of length r from this point to each point in the original space, where the line segments going to two points overlap in a line segment of distance $r - |a-b|_p$. Thus, $\mathbf{B}^1(0,r)$ is something like a tree. We will see later when we discuss reductions that adding the point $|\ |_{E(0,r)}$ is like adding a generic point to a variety in algebraic geometry.

To each point x in $\mathbf{B}^1(0,r)$, one can associate a prime ideal P_x in $T^1(r)$ by defining P_x to be the kernel of the semi-norm corresponding to x. Thus, $T^1(r)/P_x$ is an integral domain with a valuation induced by x. One denotes by $\mathcal{K}(x)$ the completion of the field of fractions of $T^1(r)/P_x$ with respect to this valuation. The image of an element $f \in T^1(r)$ in $\mathcal{K}(x)$ is denoted by $f(x)$. One then has the following result describing the fields $\mathcal{K}(x)$, where $\widetilde{\mathcal{K}(x)}$ denotes the residue class field.

Proposition 1.3. *Let x be a point in $\mathbf{B}^1(0,r)$. Let $\mathbf{F}_p^a$ denote the algebraic closure of the finite field with p elements, and let z be transcendental over $\mathbf{F}_p^a$. Then,*

(i) *If x is of type (1), then $\mathcal{K}(x) = \mathbf{C}_p$.*

(ii) *If x is of type (2), then $\widetilde{\mathcal{K}(x)}$ is isomorphic to $\mathbf{F}_p^a(z)$, and*

$$|\mathcal{K}(x)| = |\mathbf{C}_p|_p.$$

(iii) *If x is of type (3), then $\widetilde{\mathcal{K}(x)}$ is isomorphic to $\mathbf{F}_p^a$, and $|\mathcal{K}(x)|$ is generated by $|\mathbf{C}_p|_p$ and ρ, where ρ is the radius of the closed disc corresponding to x.*

(iv) *If x is of type (4), then $\mathcal{K}(x)$ is an* **immediate** *extension of $\mathbf{C}_p$, which means that it neither ramifies (i.e. $|\mathcal{K}(x)| = |\mathbf{C}_p|_p$) nor extends the residue class field (i.e. $\widetilde{\mathcal{K}(x)} = \mathbf{F}_p^a$).*

Proof: Because the kernel of a point of type (1) is a maximal ideal, (i) is clear. The kernel associated to a point of type (2), (3) or (4) is the zero ideal.

In the case of a point of type (2) or (3), let $E = E(a,\rho)$ be the closed disc such that $|\ |_E$ is the semi-norm corresponding to x. If $\rho \in |\mathbf{C}_p^\times|_p$, then there exists $b \in \mathbf{C}_p^\times$ such that $|b|_p = \rho$. Therefore, the image of $f(z) = b^{-1}(z-a)$ in $\mathcal{K}(x)$ has norm 1 and hence reduces to a transcendental element over $\mathbf{F}_p^a$, whence (ii). Now, if $\rho \notin |\mathbf{C}_p^\times|_p$, then the only elements of $T^1(r)$ whose images in $\mathcal{K}(x)$ have norm 1 are those functions such that the term c_0 in the power series expansion

$$f(z) = \sum c_k(z-a)^k$$

around a has norm 1 and such that $|c_k|_p \rho^k < 1$ for $k \geq 1$. This implies that f maps to $\tilde{c_0}$ in the residue class field and hence (iii).

To show (iv), it suffices to show that for every $f \neq 0$ in $T^1(r)$, there exists an element b in $\mathbf{C}_p$ such that

$$|f - b|_{\mathcal{E}} < |f|_{\mathcal{E}},$$

where $\mathcal{E}$ is the embedded family of closed discs giving the point x. Indeed, $\mathcal{K}(x)$ could not be ramified over $\mathbf{C}_p$ because for f and b as above, $|f|_{\mathcal{E}} = |b|_p$. Similarly, $\widetilde{\mathcal{K}(x)} = \mathbf{F}_p^a$ because if $|f|_{\mathcal{E}} = 1$, and b is as above, then $|b|_p = 1$ and $\tilde{f} = \tilde{b}$.

To show one can always find such a b, first note that by Proposition 1 of Chapter I, for every $f \neq 0$ in $T^1(r)$, there are at most finitely many a in $\mathbf{C}_p$ with $|a|_p \leq r$ such that $f(a) = 0$. Also, if $f \in T^1(r)$ is written as a power series

$$f(z) = \sum_{n=0}^{\infty} c_n(a)(z-a)^n$$

around a for some a in $\mathbf{C}_p$ with $|a|_p \leq r$, and if $0 < \rho \leq r$ is such that

$$|c_0(a)|_p \leq |c_n(a)|_p \rho^n \quad \text{for all } n \geq 1,$$

then f has a zero in the closed disc of radius ρ around the point a. Therefore, for every $f \neq 0$ in $T^1(r)$, there exists a closed disc $E = E(a, \rho) \in \mathcal{E}$ such that if

$$f(z) = \sum_{n=0}^{\infty} c_n(a)(z-a)^n,$$

then

$$|c_0(a)|_p > |c_n(a)|_p \rho^n \quad \text{for all } n \geq 1.$$

Indeed, because every such f has only finitely many zeros and because the intersection of all the closed discs in $\mathcal{E}$ was assumed to be empty, such a disc $E \in \mathcal{E}$ must exist.

Let $f \neq 0$ in $T^1(r)$ and let

$$f(z) = c_0(a) + c_1(a)(z-a) + c_2(a)(z-a)^2 + c_3(a)(z-a)^3 + \dots$$

be the power series expansion of f around a, where $|a|_p \leq r$. Notice that the power series expansion about another such point a' is given by

$$\begin{aligned}
f(z) &= c_0(a') + c_1(a')(z-a') + c_2(a')(z-a')^2 + c_3(a')(z-a')^3 + \dots \\
&= (c_0(a) + c_1(a)(a'-a) + c_2(a)(a'-a)^2 + c_3(a)(a'-a)^3 + \dots) \\
&+ (c_1(a) + 2c_2(a)(a'-a) + 3c_3(a)(a'-a)^2 + \dots)(z-a') \\
&+ (c_2(a) + 3c_3(a)(a'-a) + \dots)(z-a')^2 \\
&+ (c_3(a) + \dots)(z-a')^3 \\
&\vdots
\end{aligned}$$

In fact, it is not hard to see that

$$c_m(a') = \sum_{n=m}^{\infty} c_n(a) q_{m,n}(a'-a)^n,$$

where $q_{m,n} \in \mathbf{Z}$ and $q_{m,n} = 1$ if $m = 0$ or $n = m$.

Now the proof of (iv) can be completed. Let $f \neq 0$ be in $T^1(r)$. From the above, there exists a closed disc $E = E(a, \rho) \in \mathcal{E}$ such that

$$|c_0(a)|_p > |c_n(a)|_p \, \rho^n \quad \text{for all } n \geq 1.$$

Now, let $E(a', \rho') \subset E(a, \rho)$ be another closed disc in $\mathcal{E}$. From the above expression for $c_m(a')$ as a power series in $(a' - a)$ and since $|a' - a|_p \leq \rho$, one sees that

$$|c_n(a')|_p \leq |c_n(a)|_p \quad \text{for all } n \geq 1, \qquad \text{and} \qquad |c_0(a')|_p = |c_0(a)|_p.$$

In fact, one can say more. From the expansion of $c_0(a')$, one can conclude that

$$|c_0(a') - b|_p = |c_0(a) - b|_p$$

for all $b \in \mathbf{C}_p$ such that

$$|c_0(a) - b|_p > |c_n(a)|_p \, \rho^n \text{ for all } n \geq 1.$$

Now, let

$$0 < \delta < |c_0(a)|_p - \sup_{n \geq 1}\{|c_n(a)|_p \, \rho^n\}.$$

Because $|\mathbf{C}_p^{\times}|_p$ is dense in $\mathbf{R}_{>0}$, there exists an element b of $\mathbf{C}_p$ such that

$$\sup_{n \geq 1}\{|c_n(a)|_p \, \rho^n\} < |c_0(a) - b|_p < |c_0(a)|_p - \delta.$$

Now, if $E(a', \rho') \subset E(a, \rho)$ is another closed disc in $\mathcal{E}$, the statements above imply

$$\sup_{n \geq 1}\{|c_n(a')|_p \, \rho^n\} < |c_0(a') - b|_p < |c_0(a')|_p - \delta = |c_0(a)|_p - \delta.$$

Finally, this implies

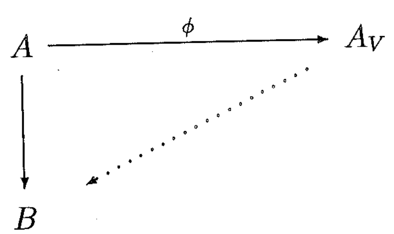
$$\begin{aligned} |f - b|_{\mathcal{E}} &= \inf_{E(a,\rho) \in \mathcal{E}} \sup_{n \geq 1}\{|c_0(a) - b|_p, |c_n(a)|_p \, \rho^n\} \\ &< \inf_{E(a,\rho) \in \mathcal{E}} \sup_{n \geq 1}\{|c_0(a)|_p, |c_n(a)|_p \, \rho^n\} = |f|_{\mathcal{E}}, \end{aligned}$$

and the proof of Proposition 1.3 is complete.

Unfortunately, the n-dimensional ball $\mathbf{B}^n(0,r)$ cannot be so easily described.

2. Affinoid Spaces. A commutative Banach $\mathbf{C}_p$-algebra A is said to be **affinoid** if there exists a surjective homomorphism $T^n(\vec{r}) \to A$. If such a homomorphism can be found with $\vec{r} = (1,\ldots,1)$, then A is said to be **strictly affinoid**. It turns out that if A is affinoid with $\vec{r} = (r_1,\ldots,r_n)$, and all $r_j \in |\mathbf{C}_p|_p$, then A is strictly affinoid. Note that the definition of "affinoid" given in [**BGR**] corresponds to what I, following [**Ber**], have called "strictly affinoid." We will not be concerned with the distinction between strictly affinoid and affinoid. Strictly affinoid algebras have some properties that affinoid algebras do not have, but Berkovich considers the larger category of affinoid algebras as a technical convenience. Everything we consider below can be assumed to be strictly affinoid.

Let A be an affinoid algebra, and let $X = \mathcal{M}(A)$. Just as before, given $x \in X$, define $\mathcal{K}(x)$ to be the completion of the fraction field of A modulo the prime ideal which is the kernel of the semi-norm $|\ |_x$ associated to x. A closed subset V in X is called an **affinoid domain** in X if there exists a bounded homomorphism of affinoid algebras $\phi: A \to A_V$ satisfying the following universal property. Given a bounded homomorphism of affinoid algebras $A \to B$ such that the image of $\mathcal{M}(B)$ in X lies in V, there exists a unique bounded homomorphism $A_V \to B$ making



commutative.

The next three propositions list some important properties of affinoid subdomains.

Proposition 2.1. *Let $X = \mathcal{M}(A)$ and let V be an affinoid subdomain of X. Let A_V and ϕ be as above. Then,*
(a) A_V and ϕ are unique up to isomorphism,
(b) the map from $\mathcal{M}(A_V)$ into X induced by ϕ is injective,
(c) and the image of $\mathcal{M}(A_V)$ under this map is exactly V.
Therefore, one can identify V and $\mathcal{M}(A_V)$.

Proof: See [**Ber**] Proposition 2.24 and [**BGR**] 7.2.2/1.

Proposition 2.2. *Let U be an affinoid subdomain of X and let V be an affinoid subdomain of U, then V is also an affinoid subdomain of X.*

Proof: This is an immediate consequence of the universal property defining affinoid subdomain.

Proposition 2.3. *Let U and V be affinoid subdomains of X, then $U \cap V$ is also an affinoid subdomain of X.*

Proof: See [**BGR**] 7.2.2/5.

Of course, unions (even finite unions) of affinoid subdomains need not be affinoid subdomains. However, a set V which is a **finite** union of affinoid subdomains is called a **special** subdomain.

One of the annoying aspects of p-adic analysis is the technical difficulty involved in getting the sheaf theory to work, or in other words to define analytic functions in such a way that they have reasonable analytic continuation properties. This process involves something called a Grothendieck topology, or G**-topology.** In fact, the subject involves several different G-topologies. I do not want to get too bogged down in these technicalities, but I will give the basic definitions.

Definition. *A G-topology on X consists of*

(a) a system S of subsets in X, called admissible open subsets and
(b) a family $\{\mathrm{Cov}\ U\}_{U \in S}$ of systems of coverings called admissible coverings, where $\mathrm{Cov}\ U$ for $U \in S$ contains coverings $\{U_i\}_{i \in I}$ of U by sets $U_i \in S$,

subject to the following conditions:

(i) If $U, V \in S$ then $U \cap V \in S$;
(ii) If $U \in S$ then $\{U\} \in \mathrm{Cov}\ U$;
(iii) If $U \in S, \{U_i\}_{i \in I} \in \mathrm{Cov}\ U$, and $\{V_{ij}\}_{j \in J_i} \in \mathrm{Cov}\ U_i$ for all $i \in I$, then $\{V_{ij}\}_{i \in I, j \in J_i} \in \mathrm{Cov}\ U$;
(iv) If $U, V \in S, V \subset U$, and $\{U_i\}_{i \in I} \in \mathrm{Cov}\ U$, then $\{V \cap U_i\}_{i \in I} \in \mathrm{Cov}\ V$.

The point of G-topologies is that sheaf theory, including Čech cohomology, works with G-topologies. If $X = \mathcal{M}(A)$, then define the **weak** G**-topology** on X by declaring the admissible open sets to be the affinoid subdomains of X (which, by the way, are closed sets in the topology on $\mathcal{M}(A)$), and by declaring that the admissible coverings consist of **finite** coverings by affinoid domains. **Warning:** If we do not restrict ourselves to finite coverings, then we will have too many analytic functions.

Let $\mathcal{O}_X$ be the pre-sheaf on X which to each affinoid subdomain Y in X associates the affinoid algebra A_Y. The importance of Tate's work [**Ta**] is that this works. Namely,

Theorem 2.4 (Tate's Acyclicity Theorem). *Let $X = \mathcal{M}(A)$ and let $\mathcal{U}$ be an admissible covering of X. Then $\mathcal{U}$ is $\mathcal{O}_X$-acyclic. In particular, $\mathcal{O}_X$ is a sheaf for the weak G-topology on X.*

In analogy with locally ringed spaces, one defines **locally G-ringed spaces,** and a locally G-ringed space obtained in the above manner is called an **affinoid space.** If the algebra A is strictly affinoid, then the space is called a **strictly affinoid space.** The reader interested in the details here is strongly encouraged to look at Part C of [**BGR**].

The correspondence $A \mapsto \mathcal{M}(A)$ is a faithful contravariant functor from the category of affinoid algebras to the category of locally G-ringed spaces, but it is not fully faithful. Therefore, one defines a morphism in the category of affinoid spaces to be a morphism of locally G-ringed spaces

$$X = \mathcal{M}(A) \to Y = \mathcal{M}(B)$$

which comes from a **bounded** homomorphism $B \to A$. Using this definition of morphism, the category of affinoid spaces is equivalent to the category of affinoid algebras.

Later, when we discuss analytic spaces, we will need to make affinoid spaces into locally ringed spaces. We can define the structure sheaf on open subsets of an affinoid space $X = \mathcal{M}(A)$ as follows. Recall that a closed subset Y in X is called **special** if it is a finite union of affinoid subdomains of X. Let Y be a special set and let $Y = \cup Y_i$ be a covering of Y by a finite number of affinoid subdomains Y_i. Define

$$A_Y = \operatorname{Ker}\left(\prod A_{Y_i} \to \prod A_{Y_i \cap Y_j}\right).$$

Tate's Acyclicity Theorem assures us that A_Y does not depend on the choice of affinoid covering Y_i. Now, let U be an open set in X. Define the structure sheaf by

$$\mathcal{O}_X(U) = \varprojlim A_Y$$

as Y ranges over all the special sets contained in U. This then gives X the structure of a locally ringed space.

We will also need to know that products exist in the category of affinoid spaces. If $X = \mathcal{M}(A)$ and $Y = \mathcal{M}(B)$, then by $X \times Y$, one means $\mathcal{M}(A \hat{\otimes} B)$, where $\hat{\otimes}$ denotes the complete tensor product. Note that just as in algebraic geometry, the topological space associated to $X \times Y$ is not the Cartesian product of the topological spaces associated to X and Y.

3. Analytic Spaces. An analytic space should be a space which locally looks like an affinoid space. However, to do this in such a way that the sheaf theory works requires some care. There are several different approaches to this. In [**BGR**], one uses Grothendieck topologies as described above. One advantage of Berkovich's theory is that for the simple cases that concern us here, Grothendieck topologies can be avoided. However, they are still useful for comparison with the theory in [**BGR**]. Berkovich has generalized his notion of analytic space in [**Ber2**] and Peter Schneider in [**Sch**] gives another generalization of Berkovich's theory, more in analogy with scheme theory. Here I will stick to the simplest Berkovich theory because I work only over $\mathbf{C}_p$ and assume all my affinoids are strictly affinoid, and this theory is sufficient for this case.

Berkovich begins by defining the notion of a quasi-affinoid space. A **quasi-affinoid** space is a pair (U, ϕ) consisting of a locally ringed space U and an open immersion ϕ of U into an affinoid space X. A closed subset Y in U is called an **affinoid domain** in U if $\phi(Y)$ is an affinoid subdomain of X. A morphism of quasi-affinoid spaces $(U, \phi) \to (U', \phi')$ is then defined to be a morphism $\theta: U \to U'$ of locally ringed spaces such that for every pair of affinoid domains Y in U and Y' in U' with $\theta(Y)$ contained in the topological interior of Y' in U', the induced homomorphism $A_{Y'} \to A_Y$ is bounded.

Finally, analytic spaces are defined as follows. Let X be a locally ringed space. An **analytic atlas** on X is a collection of pairs (U_i, ϕ_i), called charts, satisfying:

(i) The U_i form an open cover of X.
(ii) Each ϕ_i is an open immersion of U_i in an affinoid space.
(iii) For each pair i, j, the induced morphisms of locally ringed spaces

$$\phi_j \circ \phi_i^{-1}: \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is an isomorphism of quasi-affinoid spaces.

An equivalence class of analytic atlases on X defines an **analytic structure** on X and makes X into an **analytic space.** A morphism $f: X \to Y$ of locally ringed spaces is a morphism of analytic spaces if there exist atlases (U_i, ϕ_i) and (V_i, ψ_i) of X and Y respectively so that for each i and j, the morphism

$$\psi_j \circ f \circ \phi_i^{-1}: \phi_i(U_i) \to \psi_j(V_j)$$

is a morphism of quasi-affinoid spaces.

Before going on to some examples, I give one more definition. As we have seen from Tate's Acyclicity Theorem, sheaf theory works well when affinoid spaces are covered by a *finite* number of affinoid subdomains. By an **admissible affinoid cover** $\mathcal{U}$ of an analytic space X, one means a cover $\mathcal{U}$ consisting of affinoid subspaces U such that if V is any affinoid subspace of X, then $\mathcal{U}|_V = \{U \cap V\}$ is a *finite* covering of V. Note that such coverings are *not* open coverings.

I will now describe how to construct $\mathbf{P}^1$, $\mathbf{A}^1$ and $\mathbf{A}^{1\times}$ as analytic spaces, leaving the generalization to n dimensions to the reader. To construct $\mathbf{P}^1$ glue two closed balls of radius 1 at the equator as follows. Let

$$U = \mathcal{M}(\mathbf{C}_p\!<\!z_0\!>) \qquad \text{and} \qquad V = \mathcal{M}(\mathbf{C}_p\!<\!z_1\!>).$$

Then, the affinoid subsets corresponding to the "boundary" are given by

$$U' = \mathcal{M}(\mathbf{C}_p\!<\!z_0, z_0^{-1}\!>) \qquad \text{and} \qquad V' = \mathcal{M}(\mathbf{C}_p\!<\!z_1, z_1^{-1}\!>).$$

Now just identify V' and U' via the isomorphism that sends z_0 to z_1^{-1}. Note that the points $|\ |_{E(0,1)}$, which I previously said were like a generic point, are identified under the above gluing map. This means $\mathbf{P}^1$ still has this one distinguished point.

There are several ways to create $\mathbf{A}^1$. The simplest is to just observe that

$$\mathbf{A}^1 = \bigcup \mathbf{B}^1(0, r)$$

as r runs over an increasing sequence of numbers. Note that this is an admissible covering even though it contains an infinite number of sets. Before giving another atlas for $\mathbf{A}^1$, note that the algebra

$$\mathbf{C}_p\!<\!\beta^{-1}z, \alpha z^{-1}\!>$$

where $\alpha < \beta$ is the algebra of functions converging on the annulus consisting of points a in $\mathbf{C}_p$ with

$$\alpha \le |a|_p \le \beta.$$

To get $\mathbf{A}^1$, let $\alpha > 1$,

$$\begin{aligned} U_0 &= M(\mathbf{C}_p\!<\!z_0\!>), \\ U_1 &= M(\mathbf{C}_p\!<\!\alpha^{-1}z_1, z_1^{-1}\!>), \\ U_2 &= M(\mathbf{C}_p\!<\!\alpha^{-2}z_2, \alpha z_2^{-1}\!>), \\ &\ \vdots \end{aligned}$$

Then, glue along the overlapping boundaries as before. This second covering has the property that it is **formal**, which will be defined later when reductions are discussed. The situation for $\mathbf{A}^{1\times}$ is the same except that annuli are glued in both directions.

After constructing $\mathbf{A}^n$ and $\mathbf{P}^n$, one easily sees that any algebraic variety over $\mathbf{C}_p$ can be analytified.

Analytic spaces turn out to have nice topological properties, such as

Theorem 3.1 *([Ber] 3.2.1). Every connected analytic space is arc-connected.*

The following proposition illustrates that analytic spaces can be thought of as rigid analytic spaces and vice-versa.

Proposition 3.2 *([Ber] 3.3.1). (i) There is a fully faithful functor $X \mapsto X_0$ which takes an analytic space to a rigid analytic space in the sense of Tate and*

[**BGR**] *which commutes with fiber products and extension of the ground field. (ii)* X_0 *is dense in* X. *(iii)* $\mathcal{O}(X) = \mathcal{O}(X_0)$, *and (iv)* X *is reduced, normal, Cohen-Macaulay, regular, smooth, or of dimension* n *if and only if the same property holds for* X_0.

Note that in particular, the analytification of an algebraic curve is one dimensional in the ordinary topological sense of the term. Furthermore, all GAGA theorems are true for both rigid analytic spaces and for analytic spaces in the sense of Berkovich.

Finally, as a matter of notation, if X is an analytic space, I will use $X(\mathbf{C}_p)$ to denote the points in X which correspond to maximal ideals in the affinoid algebras making up a chart of X. These points are called the $\mathbf{C}_p$ points of X and are the points $x \in X$ such that $\mathcal{K}(x) = \mathbf{C}_p$.

4. Reduction. The concept of reduction is central to p-adic analysis. Let A be a strictly affinoid algebra. The ring A will be equipped with a semi-norm called the **sup semi-norm** as follows. Let $x \in \mathcal{M}(A)$. Then, x corresponds to a semi-norm $|\ |_x$ with a prime ideal P_x for kernel. The field $\mathcal{K}(x)$ has already been defined to be the completion of the fraction field of A/P_x with respect to the norm $|\ |_x$. Given f in A, denote by $f(x)$ the image of f in $\mathcal{K}(x)$. Then, one can define $|f(x)|$ to be the norm of the image of f in $\mathcal{K}(x)$ with respect to the norm $|\ |_x$. Finally, the sup semi-norm is defined on A by

$$|f|_{\text{sup}} = \sup_{x \in \mathcal{M}(A)} |f(x)|.$$

Note that $|\ |_{\text{sup}}$ need not be multiplicative, but that $|f^n|_{\text{sup}} = |f|_{\text{sup}}^n$. The sup semi-norm also satisfies the following properties.

Proposition 4.1. *(a) Given* $f \in A$, *there exists a point* $x \in \mathcal{M}(A)$ *such that* $|f(x)| = |f|_{\text{sup}}$, *and in fact, there exists an* $x \in \mathcal{M}(A)$ *such that* $P_x = \ker |\ |_x$ *is a maximal ideal and* $|f(x)| = |f|_{\text{sup}}$. *(b) Since* $\mathbf{C}_p$ *is algebraically closed,* $|A|_{\text{sup}} = |\mathbf{C}_p|_p$.

Proof: See [**BGR**], Propositions 6.2.1/1 and 6.2.1/4 for the proof.

Now define

$$A_{\leq} = \{f \in A : |f|_{\text{sup}} \leq 1\} \quad \text{and} \quad A_{<} = \{f \in A : |f|_{\text{sup}} < 1\},$$

and then

$$\widetilde{A} = A_{\leq}/A_{<}.$$

Since the sup semi-norm is power-multiplicative, the ring $\widetilde{A}$ is reduced, which means that it has no nilpotent elements. For example, if

$$A = \mathbf{C}_p\langle z\rangle, \quad \text{and} \quad f(z) = \sum a_k z^k,$$

then

$$|f|_{\text{sup}} = \sup_k |a_k|_p, \qquad \text{and} \qquad \widetilde{A} = \mathbf{F}_p^a[z],$$

the polynomial ring over the algebraic closure of the field with p elements. Therefore if $X = \mathcal{M}(A)$ is an affinoid space, one defines the **reduction** $\widetilde{X} = \operatorname{Spec} \widetilde{A}$, which associates to each affinoid space over $\mathbf{C}_p$ an affine scheme over $\mathbf{F}_p^a$. Note that even if X is non-singular, then $\widetilde{X}$ need not be non-singular. In particular, $\widetilde{X}$ is irreducible if and only if $\widetilde{A}$ is an integral domain if and only if $|\ |_{\text{sup}}$ is multiplicative on A if and only if $|\ |_{\text{sup}}$ is a point of X if and only if $\widetilde{X}$ has a unique generic point.

Let's look at one more example. Let $r \in |\mathbf{C}_p|_p$ be such that $r > 1$. Let

$$A = \mathbf{C}_p \langle r^{-1}z, r^{-1}z^{-1} \rangle.$$

Then, $X = \mathcal{M}(A)$ is the annulus $r^{-1} \leq |z|_p \leq r$. Let $a \in \mathbf{C}_p$ be such that $|a|_p = r$. Then $|a^{-1}z|_{\text{sup}} = 1$ and $|a^{-1}z^{-1}|_{\text{sup}} = 1$. However,

$$|(a^{-1}z)(a^{-1}z^{-1})|_{\text{sup}} = |a^{-2}|_{\text{sup}} = r^{-2} < 1,$$

so $|\ |_{\text{sup}}$ is not multiplicative on A. In fact, it is not hard to see that in this case,

$$\widetilde{A} \cong \mathbf{F}_p^a[\xi, \eta]/(\xi\eta),$$

so that $\widetilde{X} = \operatorname{Spec} \widetilde{A}$ is two affine lines intersecting at a point.

Given a bounded homomorphism $\phi: A \to B$, one can show that

$$|\phi(f)|_{\text{sup}} \leq |f|_{\text{sup}},$$

so it makes sense to talk about

$$\tilde{\phi}: \widetilde{A} \to \widetilde{B}.$$

This will be used to define a **reduction map**

$$\pi: X \to \widetilde{X}.$$

We have already seen that we have a map

$$\psi: A \to \mathcal{K}(x) \qquad \text{and hence} \qquad \tilde{\psi}: \widetilde{A} \to \widetilde{\mathcal{K}(x)}.$$

Now, $\widetilde{\mathcal{K}(x)}$ is a field, so $\ker(\tilde{\psi})$ is a prime ideal in $\widetilde{A}$, which gives us our map

$$\pi: \mathcal{M}(A) \to \operatorname{Spec} \widetilde{A}.$$

An important example is when

$$A = \mathbf{C}_p\!<\!z\!> \qquad \text{and} \qquad X = \mathbf{B}^1.$$

In this case we saw that

$$\widetilde{A} = \mathbf{F}_p^a[z] \qquad \text{and} \qquad \widetilde{X} = \mathbf{A}^1_{\mathbf{F}_p^a}.$$

It is easy to check that if $x = |\ |_a$ is a point of type (1), then $\pi(x)$ is the prime ideal generated by $z - \tilde{a}$. If $x = |\ |_{E(a,\rho)}$ is a point of type (2) or (3) with $\rho < 1$, then $\pi(x)$ is also the prime ideal generated by $z - \tilde{a}$. However, if $x = |\ |_{E(0,1)} = |\ |_{\sup}$, then $\pi(x)$ is the zero ideal, which is why this point behaves like a generic point. Finally, if $x = |\ |_{\mathcal{E}}$ is a point of type (4) and if $E(a,\rho)$ is a closed disc in $\mathcal{E}$ with $\rho < 1$, then $\pi(x)$ is again the prime ideal generated by $z - \tilde{a}$.

Proposition 4.2. *Let $X = \mathcal{M}(B)$ and $Y = \mathcal{M}(A)$ be strictly affinoid spaces. Let $\phi: X \to Y$ be a morphism. Let $\pi_X: X \to \widetilde{X}$ and $\pi_Y: Y \to \widetilde{Y}$ be the canonical reduction maps. Then, the following diagram is commutative.*

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & Y \\ {\scriptstyle \pi_X}\big\downarrow & & \big\downarrow{\scriptstyle \pi_Y} \\ \widetilde{X} & \xrightarrow{\ \tilde{\phi}\ } & \widetilde{Y} \end{array}$$

Proof: Let $\phi^*: A \to B$ be the algebra homomorphism inducing ϕ, and let x be a point of X. Then, one sees that both $\tilde{\phi}(\pi_X(x))$ and $\pi_Y(\phi(x))$ correspond to the ideal in $\widetilde{A}$ generated by $\{\tilde{a} : a \in A_{\le}$ and $|a|_{\phi(x)} = |\phi^*(a)|_x < 1\}$, and so the diagram commutes.

Let A be a strictly affinoid algebra. Given an ideal I in $\widetilde{A}$, denote by $V(I)$ the Zariski closed set of all prime ideals in $\widetilde{A}$ containing I, and given $\tilde{f}$ in $\widetilde{A}$, denote by $D(\tilde{f})$ the Zariski open set of all prime ideals in $\widetilde{A}$ which do not contain $\tilde{f}$. Then, the following lemma follows directly from the definitions.

Lemma 4.3. (i) *Let $\tilde{f}_j$ be generators for an ideal I in $\widetilde{A}$, with f_j in $A^{\le}$. Then,*

$$\pi^{-1}(V(I)) = \{x \in \mathcal{M}(A) : |f_j(x)| < 1 \textit{ for all } j\}.$$

(ii) *If f is in $A_{\le}$ and $|f|_{\sup} = 1$, then the set*

$$\pi^{-1}(D(\tilde{f})) = \{x \in \mathcal{M}(A) : |f(x)| = 1\}$$

is non-empty.

Because the sets $V(I)$ and $D(\tilde{f})$ generate the topology on Spec $\widetilde{A}$, this lemma implies that the map π has an anti-continuity property, namely

Corollary 4.4. *The inverse image under*

$$\pi\colon \mathcal{M}(A) \to \text{Spec } \widetilde{A}$$

of a Zariski open (resp. closed) subset of Spec $\widetilde{A}$ is an analytic closed (resp. open) subset of $\mathcal{M}(A)$. Thus, if $\mathcal{M}(A)$ is connected, Spec $\widetilde{A}$ is also connected.

Proposition 4.5 *(***[Ber]** *2.4.4). Let $X = \mathcal{M}(A)$, for a strictly affinoid algebra A, and let $\widetilde{X} =$ Spec $\widetilde{A}$. Let $\pi\colon X \to \widetilde{X}$ be the reduction map. Then, the reduction map π is surjective. If $x \in X$ and $\pi(x) = \tilde{x}$, then $\mathcal{K}(\tilde{x}) \subseteq \widetilde{\mathcal{K}(x)}$. Furthermore, if $\tilde{x}$ is a generic point of $\widetilde{X}$, then there exists a unique point x in X such that $\pi(x) = \tilde{x}$. For such points, one has $\widetilde{\mathcal{K}(x)} \cong \mathcal{K}(\tilde{x})$, where $\mathcal{K}(\tilde{x})$ is the function field associated to $\tilde{x}$.*

Proof: The proof of the fact that π is surjective will be omitted. See **[BGR]** 7.1.5/4. That $\mathcal{K}(\tilde{x}) \subseteq \widetilde{\mathcal{K}(x)}$ follows directly from the definitions.

To prove the statement about the generic point, first assume that $\widetilde{X}$ is irreducible. In this case the sup semi norm is multiplicative and therefore corresponds to a point of X. Let x be a point in $\pi^{-1}(\tilde{x})$. We will show that

$$|f(x)| = |f|_{\text{sup}} \quad \text{for all } f \in A,$$

and therefore that $|\ |_x = |\ |_{\text{sup}}$. If $|f|_{\text{sup}} = 0$, then $f^n = 0$ for some n, so $|f(x)|$ is also 0. Because of the assumption that A is strictly affinoid, if $|f|_{\text{sup}} > 0$, then there exists an $a \in \mathbf{C}_p^\times$ such that $|f|_{\text{sup}} = |a|_p$ by Proposition 4.1. Thus, let $g = a^{-1}f$ so that $|g|_{\text{sup}} = 1$. Since $\widetilde{X}$ is irreducible, this implies that $\tilde{g}$ is not contained in the prime ideal associated to the generic point $\tilde{x}$. Therefore, $|g(x)| = 1$, and hence $|f(x)| = |f|_{\text{sup}}$.

For reducible $\widetilde{X}$, let g be an element in $A_{\leq}$ such that $\tilde{g}(\tilde{x}) \neq 0$ but such that $\tilde{g}$ vanishes on all the components of $\widetilde{X}$ not containing $\tilde{x}$. Then, the set

$$\{x \in X : |g(x)| = 1\}$$

is affinoid equal to $\mathcal{M}(B)$, where $B = A < g^{-1} >$, the set of power series in g^{-1} with coefficients in A whose sup norms are tending toward zero. Then since

$$\pi^{-1}(\tilde{x}) \subset \mathcal{M}(B)$$

and Spec $\widetilde{B}$ is irreducible, the general case follows from the irreducible case. In other words,

$$|f|_x = \sup\{|f(x)| : x \text{ such that } |g(x)| = 1\}.$$

To show that $\widetilde{\mathcal{K}(x)} \cong \mathcal{K}(\tilde{x})$, it suffices to assume that A is reduced. First assume that $\widetilde{A}$ is integral. Then, $\mathcal{K}(\tilde{x})$ is the function field of $\widetilde{A}$. Since the function field of A is dense in $\mathcal{K}(x)$, any element in $\widetilde{\mathcal{K}(x)}$ is the image of f/g for some $f, g \in A$ with $|f|_{\text{sup}} = |g|_{\text{sup}}$. By Proposition 4.1, there exists an $a \in \mathbf{C}_p$ such that $|a|_p = |f|_{\text{sup}} = |g|_{\text{sup}}$, so any element of $\widetilde{\mathcal{K}(x)}$ is the image of f'/g', where $|f'|_{\text{sup}} = |g'|_{\text{sup}} = 1$. This proves the statement when $\widetilde{A}$ is an integral domain. The general case follows from this case as above, and therefore the proof of Proposition 4.5 is now complete.

Let $X = \mathcal{M}(A)$ be an affinoid space. For an arbitrary point $\tilde{x} \in \widetilde{X}$, determining what $\pi^{-1}(\tilde{x})$ looks like is a difficult problem. However if $\tilde{x}$ is a smooth closed point, and X is n-dimensional, then this inverse image looks like the "open" unit ball

$$\mathring{\mathbf{B}}^n = \{x \in \mathbf{B}^n : |z_j|_x < 1 \text{ for } j = 1, \ldots, n\}.$$

Proposition 4.6 ([BL1] Proposition 2.2). *Let $X = \mathcal{M}(A)$ be a strictly affinoid space. Let $\tilde{x}$ be a closed point in $\widetilde{X}$, and let $U = \pi^{-1}(\tilde{x})$. If y is a point in the closure of U such that $\pi(y)$ is a closed point in $\widetilde{X}$, then y is a point in U. If, in addition, $\tilde{x}$ is a smooth closed point, and X is pure n-dimensional, then there exists an analytic map*

$$X \to \mathbf{B}^n$$

which restricts to an analytic isomorphism

$$U \to \mathring{\mathbf{B}}^n$$

Thus, if X is one dimensional and $\tilde{x}'$ is the generic point of $\widetilde{X}$ on the irreducible component containing $\tilde{x}$, then the closure of U in X is $U \cup \{x'\}$, where x' is the unique point lying above $\tilde{x}'$. (The point x' is unique by Proposition 4.5.)

Proof: Let y be a point in the closure of U such that $\tilde{y}$ is closed and not equal to $\tilde{x}$. Now,

$$F = \pi^{-1}(\widetilde{X} - \{\tilde{y}\})$$

is a closed set in X by Corollary 4.4 since $\widetilde{X} - \{\tilde{y}\}$ is Zariski open in $\widetilde{X}$. But, F contains U and not y, so y cannot be in the closure of U, and thus all points in the closure of U which lie above closed points must in fact be in U.

Now, assume that $\tilde{x}$ is smooth and that X is pure n-dimensional. Let $M_{\tilde{x}}$ be the maximal ideal in $\widetilde{A}$ corresponding to $\tilde{x}$. Since $\tilde{x}$ is smooth, there exist n elements $f_1, \ldots, f_n \in A_{\leq}$ such that $\tilde{f}_1, \ldots, \tilde{f}_n$ generate $M_{\tilde{x}}$.

Now, as an $\mathbf{F}_p^a$ module,

$$\widetilde{A} = \mathbf{F}_p^a \oplus M_{\tilde{x}} = \mathbf{F}_p^a \oplus \widetilde{A}\tilde{f}_1 + \cdots + \widetilde{A}\tilde{f}_n.$$

One can then show (see [**BL1**] Proposition 2.2 and the references cited therein) that this implies

$$(*) \qquad A_{\leq} = (\mathbf{C}_p)_{\leq} + A_{\leq} f_1 + \cdots + A_{\leq} f_n,$$

where $(\mathbf{C}_p)_{\leq}$ denotes the set of $a \in \mathbf{C}_p$ with $|a|_p \leq 1$.

Next, define a map from $\mathbf{C}_p < z_1, \ldots, z_n >$ to A by sending $z_j \mapsto f_j$. Because $|f_j|_{\text{sup}} = 1$, this map is well-defined and induces an analytic map $X \to \mathbf{B}^n$. The restriction of this map to

$$U_\varepsilon = \{x \in X : |f_j(x)| \leq \varepsilon \text{ for all } j\} = \mathcal{M}(A < \varepsilon^{-1} f_1, \ldots, \varepsilon^{-1} f_n >),$$

where $\varepsilon < 1$ is given by the map

$$\phi_\varepsilon : \mathbf{C}_p < \varepsilon^{-1} z_1, \ldots, \varepsilon^{-1} z_n > \rightarrowtail A < \varepsilon^{-1} f_1, \ldots, \varepsilon^{-1} f_n >,$$

defined by $z_j \mapsto f_j$.

Now, ϕ_ε is injective by reason of dimension. The idea it to show that ϕ_ε is surjective (for $\varepsilon < 1$) by using $(*)$ inductively, and to do this it suffices to show that any element of $A_{\leq}$ is in the image of ϕ_ε.

But, if $g \in A_{\leq}$, then

$$g = c_0 + \sum_{j=1}^{n} a_{0,j} f_j,$$

where $c_0 \in (\mathbf{C}_p)_{\leq}$ and $a_{0,j} \in A_{\leq}$. But then,

$$a_{0,j} = c_{1,j} + \sum_{k=1}^{n} a_{1,j,k} f_k,$$

so

$$g = c_0 + \sum_{j=1}^{n} c_{1,j} f_j + \sum_{j=1}^{n} \sum_{k=1}^{n} a_{1,j,k} f_j f_k.$$

Continuing this process, we see that g can be written as a formal power series in $f_1, \ldots, f_n$ with coefficients in $(\mathbf{C}_p)_{\leq}$. Because $\varepsilon < 1$, g is in the image of ϕ_ε, and therefore ϕ_ε is an isomorphism when $\varepsilon < 1$.

Because $U = \bigcup_{\varepsilon < 1} U_\varepsilon$, one sees that

$$U \to \mathring{\mathbf{B}}^n$$

is an isomorphism, and the proof of the theorem is complete.

I would now like to briefly discuss the reductions of analytic spaces which are not affinoid. However, analytic spaces which are not affinoid do not have canonical reductions, so let's look at $\mathbf{P}^1$ to get an idea of what to do in the general case. Recall that $\mathbf{P}^1$ was defined as a union of affinoid sets

$$U_0 = \mathcal{M}(\mathbf{C}_p \langle z_0 \rangle) \qquad \text{and} \qquad U_1 = \mathcal{M}(\mathbf{C}_p \langle z_1 \rangle),$$

with $U_j' = \mathcal{M}(\mathbf{C}_p \langle z_j, z_j^{-1} \rangle)$ identified via $z_0 \mapsto z_1^{-1}$. Let

$$\pi_j : U_j \to \widetilde{U}_j \cong \mathbf{A}^1_{\mathbf{F}_p^a}$$

be the reduction maps. Note that

$$\pi_j(U_j') \cong \mathbf{A}^1_{\mathbf{F}_p^a} - \{0\},$$

which is Zariski open, and that the gluing upstairs gives rise downstairs to the standard gluing of two affine lines to get $\mathbf{P}^1_{\mathbf{F}_p^a}$.

The general case is as follows. Let U be a strictly affinoid space. Let V be a strictly affinoid subdomain of U. If the induced morphism $\widetilde{V} \to \widetilde{U}$ is an open immersion, then V is called a **formal affinoid subdomain** of U. Now let X be a separated analytic space. An admissible, strictly affinoid covering $\mathcal{U}$ of X is called **formal** if the intersection $U \cap V$ is a formal strictly affinoid subdomain of U for every $U, V \in \mathcal{U}$. Given a formal covering $\mathcal{U}$ of X, one gets an algebraic variety $\widetilde{X_\mathcal{U}}$, locally of finite type over $\mathbf{F}_p^a$, and a reduction map

$$\pi_\mathcal{U} : X \to \widetilde{X_\mathcal{U}}.$$

If $\mathcal{U}$ and $\mathcal{V}$ are two formal affinoid coverings of X, then the reduced varieties $\widetilde{X_\mathcal{U}}$ and $\widetilde{X_\mathcal{V}}$ are in general non-isomorphic. However, two formal coverings $\mathcal{U}$ and $\mathcal{V}$ are called equivalent if $U \cap V$ is a finite union of formal subdomains of both U and V for every $U \in \mathcal{U}$ and every $V \in \mathcal{V}$. Equivalent formal coverings give rise to isomorphic reductions.

Remark. When working over non-algebraically closed fields, one also wants a technical condition known as distinguishedness. Since $\mathbf{C}_p$ is algebraically closed, this will not concern us here.

Caution! If $\phi: X \to Y$ is a morphism of analytic spaces, $\mathcal{U}$ is a formal cover of X and $\mathcal{V}$ is a formal cover of Y, then ϕ does not necessarily induce a morphism $\tilde{\phi}: \widetilde{X_\mathcal{U}} \to \widetilde{Y_\mathcal{V}}$. However, if there exists a formal cover $\mathcal{U}'$ of X equivalent to $\mathcal{U}$ and a formal cover $\mathcal{V}'$ of Y equivalent to $\mathcal{V}$ such that for every $U \in \mathcal{U}'$ there exists a $V \in \mathcal{V}'$ such that $\phi(U) \subset V$, then ϕ does induce a morphism $\tilde{\phi}: \widetilde{X_\mathcal{U}} \to \widetilde{Y_\mathcal{V}}$, and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad \phi \quad} & Y \\ \Big\downarrow{\scriptstyle \pi_{\mathcal{U}}} & & \Big\downarrow{\scriptstyle \pi_{\mathcal{V}}} \\ \widetilde{X_{\mathcal{U}}} & \xrightarrow{\quad \tilde{\phi} \quad} & \widetilde{Y_{\mathcal{V}}} \end{array}$$

commutes.

5. Normalizations. In this section we introduce the concept of the "normalization" of an analytic space. Normalizations are a kind of desingularization of analytic spaces. They will be most useful for one dimensional analytic spaces precisely because the normalization of a one dimensional analytic space is non-singular. *Note that all analytic spaces are assumed to be reduced.*

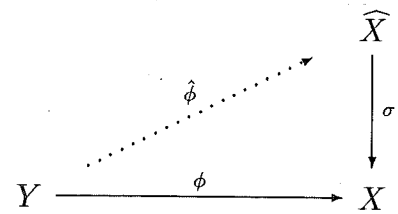
We begin with some definitions. Let X be a reduced analytic space. A point x in X is said to be a **normal point** of X if the local ring at the point $\mathcal{O}_{X,x}$ is integrally closed (in its field of fractions). An analytic space X is called **normal** if all of its points are normal points. Recall also that a closed subset T of an analytic space is said to be **thin of order** $n \geq 1$ if every point x in X has an open neighborhood U such that $T \cap U$ is contained in a nowhere dense closed analytic subset Y of U such that

$$\dim_y(Y) \leq \dim_y(X) - n \quad \text{for all points } y \text{ in } Y.$$

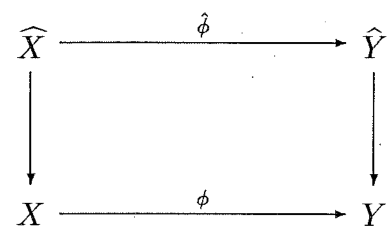
All we need are the following facts about normalizations. See [**Ber**] Section 3.3, especially Corollary 3.3.18 for partial statements of these results. See [**Lü**] for the proof of the non-Archimedean version of the Riemann Extension Theorem, which is necessary for the proofs of the results stated in [**Ber**]. Finally, see Chapter 8 of [**GR**] for the classical treatment of this topic for complex analytic spaces.

Fact 5.1. *Let X be a reduced normal analytic space. Then, the set $S(X)$ of singular points in X is thin of order 2 in X.*

Fact 5.2 (Existence of Normalizations). *Let X be a reduced analytic space. Let $N(X)$ be the set of non-normal points in X. There exists a unique normal analytic space $\widehat{X}$ and a morphism $\sigma\colon \widehat{X} \to X$ satisfying the following universal property. Given any reduced normal analytic space Y and a morphism $\phi\colon Y \to X$ such that the set $\phi^{-1}(N(X))$ is nowhere dense in Y, there exists a unique morphism $\hat{\phi}$ making the following diagram commutative.*



Fact 5.3 (Map Lifting). *Let $\phi: X \to Y$ be a morphism of reduced analytic spaces such that the inverse image $\phi^{-1}(N(Y))$ of the set of non-normal points in Y is nowhere dense in X. Let $\widehat{X}$ and $\widehat{Y}$ be the normalizations of X and Y respectively. Then, there exists a unique morphism $\hat{\phi}$ making the following diagram commutative, where the vertical arrows are the normalization maps.*



Fact 5.4. *If X is a reduced analytic space, then the normalization map $\sigma: \widehat{X} \to X$ is such that if $N(X)$ is the set of non-normal points in X, then*

$$\sigma: \widehat{X} \setminus \sigma^{-1}(N(X)) \to X \setminus N(X)$$

is an isomorphism.

CHAPTER V

Algebraic Curves

In this chapter we see how Berkovich applies his theory to the study of algebraic curves. Berkovich uses this theory to prove that any analytic map from $\mathbf{A}^1$ into a curve of genus ≥ 1 is constant. In this chapter, I explain Berkovich's proof and point out that it also shows that any analytic map from $\mathbf{A}^{1\times}$ into a smooth projective curve of genus ≥ 2 or into an elliptic curve with good reduction is constant.

1. One Dimensional Analytic Spaces. We begin by noting that as a topological space, $\mathbf{P}^1$ has the following nice property.

Theorem 1.1 *([**Ber**] 4.2.1). $\mathbf{P}^1$ is a simply connected special quasi-polyhedron. In particular, all of the connected locally compact subsets of $\mathbf{P}^1$ are contractible.*

The concept of a quasi-polyhedron is not important here; see [**Ber**] for its definition. Note that since we saw that $\mathbf{P}^1$ is something like a tree, this result is not too surprising.

Let $\mathbf{A}^{1\times}$ denote the analytic space corresponding to the affine line minus the origin in the sense of algebraic geometry. Since

$$\mathbf{A}^{1\times} = \bigcup_{r>1} \mathcal{M}(\mathbf{C}_p\!<\!r^{-1}z, r^{-1}z^{-1}\!>),$$

$\mathbf{A}^{1\times}$ is locally compact and connected. Hence by the above proposition, $\mathbf{A}^{1\times}$ is contracible.

We also have the following result about curves of higher genus.

Theorem 1.2 *([**Ber**] 4.3.3). Any one dimensional analytic space is locally contractible. In particular, if it is separated and connected, then it has a universal covering.*

2. Semi-Stable Reduction Theorem. We remarked at the end of the last chapter that in general analytic spaces do not have canonical reductions. However, one does have the following existence theorem for curves, which says one can find a formal affinoid covering whose associated reduction has at most mild singularities, called ordinary double points. An **ordinary double point** on an algebraic curve over $\mathbf{F}_p^a$ is a point such that the completion of the local ring at the point is isomorphic to

$$\mathbf{F}_p^a[[\xi, \eta]]/(\xi\eta).$$

Also, recall that an algebraic curve is called **rational** if it is birational to $\mathbf{P}^1$.

Let X be the analytification of a smooth projective algebraic curve over $\mathbf{C}_p$. The space X is said to have **semi-stable reduction** if there exists a formal affinoid covering $\mathcal{U}$ of X such that

(a) $\widetilde{X_{\mathcal{U}}}$ is reduced and
(b) $\widetilde{X_{\mathcal{U}}}$ has at worst ordinary double points for singularities.

In addition, one often adds the condition

(c) If X has genus ≥ 1, then each non-singular rational irreducible component on $\widetilde{X_{\mathcal{U}}}$ meets the other components in at least two points.

If X has genus ≥ 2 and if there exists a formal affinoid covering $\mathcal{U}$ of X satisfying (a) and (b), plus the additional condition,

(c') Every non-singular rational irreducible component of $\widetilde{X_{\mathcal{U}}}$ meets the other components in at least three points,

then X is said to have **stable reduction.** Note that the analytic reduction $\widetilde{X_{\mathcal{U}}}$ is always reduced, so condition (a) is superfluous here. The condition is made explicit in analogy with the scheme theoretic definition over non-algebraically closed fields, where reducedness is not automatic.

Theorem 2.1 (Semi-Stable Reduction Theorem) [BL1]. *Let X be the analytification of a smooth connected projective algebraic curve. Then,*

(a) If X has genus 0, then there exists a formal covering $\mathcal{U}$ such that $\widetilde{X_{\mathcal{U}}} \cong \mathbf{P}^1$.
(b) If X has genus 1, then X has semi-stable reduction. Furthermore, if $\widetilde{X_{\mathcal{U}}}$ is non-singular of genus 1 for some formal affinoid covering $\mathcal{U}$ giving a semi-stable reduction, then $\widetilde{X_{\mathcal{V}}} \cong \widetilde{X_{\mathcal{U}}}$ for all formal affinoid coverings $\mathcal{V}$ giving semi-stable reductions;
(c) If X has genus ≥ 2, then X has stable reduction and all stable reductions of X are isomorphic.

Moreover, if $\widetilde{X_{\mathcal{U}}}$ is a reduction as above, then

$$\dim H^1(\widetilde{X_{\mathcal{U}}}, \mathcal{O}_{\widetilde{X_{\mathcal{U}}}}) = \dim H^1(X, \mathcal{O}_X) = g,$$

where g is the genus of X.

Remarks. Even though semi-stable reductions are not unique, if X has genus 1, then either $\widetilde{X_{\mathcal{U}}}$ is non-singular, or all of its components are rational; this dichotomy does not depend on the covering used to obtain the reduction. If X has genus ≥ 2 (resp. $= 1$) and all of the components of the stable (resp. a semi-stable) reduction are rational, then X is said to have **totally degenerate reduction.** We will see that curves with totally degenerate reduction have a nice uniformization theory.

Note that the above theorem is proved in [**BL**] by purely p-adic analytic means and does not need the scheme theoretic version of the theorem. In particularly, [**BL1**] and [**BL2**] do not need any assumptions about the ground field being discretely valued. For more details on reductions see [**BGR**] and [**BL1**].

3. Uniformization. Let Γ be a subgroup of $PGL_2(\mathbf{C}_p)$. A point x in $\mathbf{P}^1(\mathbf{C}_p)$ is called a **limit point** of Γ if there exists a point y in $\mathbf{P}^1(\mathbf{C}_p)$ and an infinite sequence $\{\gamma_n\}_{n\geq 0} \subset \Gamma$ such that

$$\lim_{n\to\infty} \gamma_n(y) = x.$$

Let Σ_Γ denote the set of limit points of Γ. The group Γ is called **discontinuous** if the set of limit points Σ_Γ is not equal to all of $\mathbf{P}^1(\mathbf{C}_p)$, and if for any point $x \in \mathbf{P}^1(\mathbf{C}_p)$, the closure of the orbit of x under Γ in $\mathbf{P}^1(\mathbf{C}_p)$ is compact. The group Γ is called a **Schottky group** if

(a) Γ is finitely generated,
(b) Γ has no non-trivial elements of finite order and
(c) Γ is discontinuous.

For Schottky groups, one can show if Γ has rank 1, then Σ_Γ consists of two points, and if Γ has rank ≥ 2, then Σ_Γ contains infinitely many points; this will be important later.

It turns out that a Schottky group Γ acts freely on $\Omega_\Gamma = \mathbf{P}^1 - \Sigma_\Gamma$, where here $\mathbf{P}^1$ is the analytic space, not just the $\mathbf{C}_p$ points. The quotient space Ω_Γ/Γ is compact, and therefore by the appropriate GAGA theorem, it is the analytification of a smooth geometrically irreducible projective curve X. Such a curve is called a **Mumford curve.** For more information about Schottky groups and Mumford curves, see [**G-P**]. The uniformization theory of Mumford curves is summarized by the following theorem.

Theorem 3.1 *([**Ber**] 4.4.1). The following properties of a smooth geometrically connected projective curve X of genus $g \geq 1$ are equivalent.*

(a) X is a Mumford curve.
(b) The universal covering Ω of X is isomorphic to an open subset of $\mathbf{P}^1$ whose complement lies in $\mathbf{P}^1(\mathbf{C}_p)$.
(c) X has totally degenerate reduction.

Remark. Since the genus of a Mumford curve is equal to the rank of the Schottky group giving the curve, if the genus is larger than one, then the complement of its universal cover in $\mathbf{P}^1$ consists of an infinite number of $\mathbf{C}_p$ points.

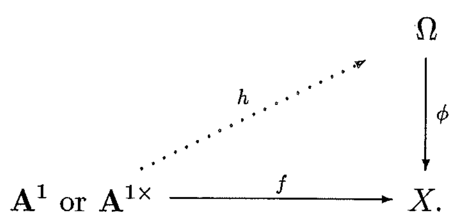
4. Hyperbolicity. Now I give Berkovich's proof of the fact that curves of genus ≥ 2 are p-adic Brody hyperbolic.

Theorem 4.1. *Let X be an irreducible algebraic curve defined over $\mathbf{C}_p$. Assume that X has (geometric) genus ≥ 1. If $f\colon \mathbf{A}^1 \to X$ is an analytic map into X, then f is constant. Similarly, if $f\colon \mathbf{A}^{1\times} \to X$ is an analytic map, then either f is constant, or X is a projective curve whose normalization is an elliptic curve with bad reduction.*

Proof: The proof of this theorem demonstrates a general philosophy in p-adic analytic geometry. The basic idea is to reduce the problem mod p, and then one of two things will happen. Either the problem will retain a sufficient amount of its geometric character mod p, and algebraic geometry over $\mathbf{F}_p^a$ can be used to solve the problem, or the geometry mod p will degenerate so much that some special technique, in this case uniformization, can be applied to the analytic problem.

Since X is a subset of an irreducible projective curve, we may assume that X is projective and irreducible. For now, we will also assume that X is smooth. By the Semi-Stable Reduction Theorem (Theorem 2.1), the analytic curve X admits a semi-stable reduction $\widetilde{X}$, which we may assume stable if $g \geq 2$.

First, we will treat the case when the problem mod p retains none of its geometric character, and this is precisely the case where we will be able to exploit the uniformization theorem of the last section. Suppose that all of the components of $\widetilde{X}$ are rational (i.e. that X has totally degenerate reduction). Then, by Theorem 3.1, X is a Mumford curve, and there is a uniformization $\phi\colon \Omega \to X$, where Ω is an open subset of $\mathbf{P}^1$. Now, the complement of Ω contains an infinite number of $\mathbf{C}_p$ points if X has genus ≥ 2, whereas the complement contains two points if X has genus 1. In any case, we may therefore assume that Ω lies in $\mathbf{A}^{1\times}$, and does not coincide with all of $\mathbf{A}^{1\times}$ if X has genus larger than one. Furthermore, since $\mathbf{A}^1$ and $\mathbf{A}^{1\times}$ are contractible, there exists a commutative diagram of analytic maps



If X has genus ≥ 2, then h is constant by Proposition 4 of Chapter I since it is an analytic function on $\mathbf{C}_p$ or $\mathbf{C}_p^{\times}$ omitting at least two values. If X has genus 1 and we are in the case that f is a map from $\mathbf{A}^1$ into X, then h is also constant by Proposition 4 of Chapter I since it an analytic function on $\mathbf{C}_p$ without a zero. Also, if f is a non-constant map from $\mathbf{A}^{1\times}$ into X and X has genus 1, then since we are in the case of degenerate reduction, X must be an elliptic curve with bad reduction. Finally, note that if the original curve X were not projective, then h

would have to lift to a map whose image is properly contained in $\mathbf{A}^{1\times}$, and would therefore have to be constant.

Now, suppose $\widetilde{X}$ contains a non-rational component $\widetilde{Y}$, so that the problem mod p retains some of its geometric flavor. Let $\tilde{y}$ be the generic point of $\widetilde{Y}$, and let y be the inverse image of $\tilde{y}$ under the reduction map $\pi: X \to \widetilde{X}$. The fact that there is only one point in the inverse image of $\tilde{y}$ follows from Proposition IV.4.5. Because $\widetilde{\mathcal{K}(y)} \cong \mathcal{K}(\tilde{y})$, again by Proposition IV.4.5, one knows that y is not in the image of f. Indeed, if $y = f(z)$ for some z, then $\widetilde{\mathcal{K}(y)}$ is embedded in $\widetilde{\mathcal{K}(z)}$, but by Proposition IV.1.3, $\widetilde{\mathcal{K}(z)}$ for z in $\mathbf{A}^1$ or $\mathbf{A}^{1\times}$ is either $\mathbf{F}_p^a$ or the field of rational functions over $\mathbf{F}_p^a$.

Let $\tilde{y}'$ and $\tilde{y}''$ be two distinct closed points of $\widetilde{Y}$ which are smooth in $\widetilde{X}$. Then, from Proposition IV.4.6,

$$\overline{\pi^{-1}(\tilde{y}')} = \pi^{-1}(\tilde{y}') \cup \{y\} \quad \text{and} \quad \overline{\pi^{-1}(\tilde{y}'')} = \pi^{-1}(\tilde{y}'') \cup \{y\}.$$

Since $\mathbf{A}^1$ and $\mathbf{A}^{1\times}$ are arc-connected and y is not contained in the image of $\mathbf{A}^1$ or $\mathbf{A}^{1\times}$ under f, the open subsets $\pi^{-1}(y')$ and $\pi^{-1}(y'')$ cannot both contain points in the image of f. Hence, there exist a $\mathbf{C}_p$ point x in X and an open neighborhood U of x in X which does not meet the image of f.

Now by Riemann-Roch, there exists a rational function h on X with a pole only at the point x. Taking the composition of

$$f: \mathbf{A}^1 \text{ or } \mathbf{A}^{1\times} \to X \quad \text{with} \quad h: X - U \to \mathbf{A}^1,$$

one obtains a bounded analytic function $\mathbf{A}^1$ or $\mathbf{A}^{1\times} \to \mathbf{A}^1$, which is therefore constant.

We are left with the case that X is not smooth. Assume that we have a non-constant analytic map f into X. Since X is one-dimensional, the image of f must then contain an open set. Therefore, the inverse image under f of the singular points in X is nowhere dense, and we can apply Fact IV.5.3 to lift f to an analytic map $\hat{f}$ into the normalization $\widehat{X}$ of X. Now because $\widehat{X}$ has dimension one, Fact IV.5.1 implies that $\widehat{X}$ is smooth. The map $\hat{f}$ is hence constant by the above, and therefore f must also have been constant. This then completes the proof of the theorem.

Remark. The above proof shows that if Ω is an open subset of $\mathbf{P}^1$ without bounded functions, then there are no non-constant maps into an elliptic curve with good reduction. This means that there is no open subset Ω of $\mathbf{P}^1$ which will map into an elliptic curve with good reduction in a non-constant way, but such that the only analytic maps from Ω into $\mathbf{P}^1$ which miss three points are the constant maps. Therefore, I think it is unlikely that there is an intuitive p-adic notion of Brody hyperbolicity such that algebraic varieties X defined over $\mathbf{Q}$ would satisfy a theorem stating that X is Brody hyperbolic at p if and only if X is Brody hyperbolic at infinity.

CHAPTER VI

Analytic Tori and Abelian Varieties

In this chapter some p-adic analytic groups are examined. Section 1 gives some basic definitions, and then Section 2 describes products of the multiplicative group and quotients by discrete subgroups – such spaces are called analytic tori. Section 3 discusses the uniformization and reduction of Abelian varieties after Berkovich [**Ber**] and Bosch and Lütkebohmert [**BL2**]. Then, Section 4 applies this theory to the study of analytic maps from $\mathbf{A}^1$ and $\mathbf{A}^{1\times}$ into subvarieties of Abelian varieties. Here I show that Berkovich's argument from the previous chapter also shows that the only analytic maps from $\mathbf{A}^1$ into Abelian varieties are the constant maps. I then give a proof of a p-adic analogue of Bloch's conjecture for p-adic analytic maps from $\mathbf{A}^{1\times}$ into varieties with ample irregularity.

1. Analytic Groups. Before beginning the discussion of analytic groups, let's start with an observation about one way in which p-adic analytic spaces differ from complex analytic spaces. If X is a complex analytic space and Y is a closed complex subspace of X, then for any point $y \in Y$, there exist an analytic open neighborhood U of y in X and a finite number of analytic functions $f_1, \dots, f_n$ on U such that

$$Y \cap U = \{x \in U : f_1(x) = \cdots = f_r(x) = 0\}.$$

On the other hand, p-adic analytic spaces have too many closed analytic subspaces. For example, affinoid subspaces are closed. Therefore, we make the following definition.

Definition. *A closed analytic subspace Y of X is called an* **analytic subvariety** *of X if there exists an* **admissible** *affinoid covering $\mathfrak{U}$ of X such that for every $U \in \mathfrak{U}$, there exist a finite number of functions $f_1, \dots, f_r$ in $\mathcal{O}_X(U)$ such that*

$$Y \cap U = \{x \in U : f_1(x) = \cdots = f_r(x) = 0\}.$$

Remark. The requirement that the covering be admissible is absolutely essential. Without this condition, affinoid subspaces of projective spaces would qualify as subvarieties.

We also define the **analytic Zariski topology** on X to be the weakest topology on X such that all the analytic subvarieties of X are closed sets.

By an **analytic group** G, one means a group object in the category of analytic spaces. This means that there exist three morphisms

a) $\mu: G \times G \to G$ (multiplication)
b) $i: G \to G$ (inverse)
c) $e: G \to G$ (identity)

satisfying the obvious relations. Note that G itself is not a group, but it follows easily that $G(\mathbf{C}_p)$ is a group.

The two most important examples of analytic groups are the additive group $\mathbf{G}_a$, which as an analytic space is isomorphic to $\mathbf{A}^1$, and the multiplicative group $\mathbf{G}_m$, which as an analytic space is isomorphic to $\mathbf{A}^{1\times}$.

Proposition 1.1. *Let G be an analytic group. Then, the following morphisms $G \to G$ are homeomorphisms for the analytic Zariski topology on G:*
(a) $g \mapsto g^{-1}$
(b) $g \mapsto xg$ for a fixed $x \in G(\mathbf{C}_p)$.

Proof: Let ϕ be either of the morphisms above. Let Y be an analytic subvariety of G. Let $\mathfrak{U}$ be an admissible affinoid covering of G such that

$$Y \cap U = \{g \in U : f_1(g) = \cdots = f_r(g) = 0\},$$

for a finite number of functions $f_1, \ldots, f_r$ in $\mathcal{O}_G(U)$. Because ϕ is an isomorphism of analytic spaces (though not, in general, a group homomorphism) $\phi^{-1}(U)$ is affinoid for every U in $\mathfrak{U}$. Furthermore,

$$\phi^{-1}(Y) \cap \phi^{-1}(U) = \{g \in \phi^{-1}(U) : f_1 \circ \phi(g) = \cdots = f_r \circ \phi(g) = 0\},$$

so $\phi^{-1}(Y)$ is an analytic subvariety, and hence closed in the analytic Zariski topology.

The following lemma will be applied to an analytic group G with the analytic Zariski topology.

Lemma 1.2 ([L1] page 84). *Let G be a group with a topology. (The topology is not assumed to be Hausdorff, and G is not assumed to be a topological group, because this lemma will be applied to a topology which is not Hausdorff and such that the multiplication map is not continuous when $G \times G$ is given the product topology.) Assume that the map $g \mapsto g^{-1}$ is a homeomorphism and that for each x in G, the map $g \mapsto xg$ is also a homeomorphism. If H is an abstract subgroup of G, then the closure H^{cl} of H in G is also a subgroup of G.*

Proof: Since $g \mapsto g^{-1}$ is a homeomorphism, one has

$$H^{\mathrm{cl}} = (H^{-1})^{\mathrm{cl}} \subset (H^{\mathrm{cl}})^{-1}.$$

Applying $g \mapsto g^{-1}$ gives

$$(H^{\mathrm{cl}})^{-1} \subset H^{\mathrm{cl}}, \text{ so } H^{\mathrm{cl}} = (H^{\mathrm{cl}})^{-1}.$$

Next, let $h \in H$. Then,

$$hH \subset H^{\mathrm{cl}}, \text{ so } H \subset h^{-1}H^{\mathrm{cl}},$$

which is closed since $g \mapsto h^{-1}g$ is a homeomorphism. Therefore,

$$H^{\mathrm{cl}} \subset h^{-1}H^{\mathrm{cl}} \text{ and hence } hH^{\mathrm{cl}} \subset H^{\mathrm{cl}}.$$

Thus,

$$HH^{\mathrm{cl}} \subset H^{\mathrm{cl}} \text{ and } H^{\mathrm{cl}}H \subset H^{\mathrm{cl}}.$$

Finally, let h be an element of H^{cl}. Then,

$$hH \subset H^{\mathrm{cl}} \text{ and so } H \subset h^{-1}H^{\mathrm{cl}}.$$

Therefore,

$$H^{\mathrm{cl}} \subset h^{-1}H^{\mathrm{cl}} \text{ and so } hH^{\mathrm{cl}} \subset H^{\mathrm{cl}}.$$

Thus,

$$H^{\mathrm{cl}}H^{\mathrm{cl}} \subset H^{\mathrm{cl}},$$

so H^{cl} is a subgroup.

Corollary 1.3. *Let H be an abstract subgroup of $G(\mathbf{C}_p)$ for an analytic group G. Then, the analytic Zariski closure of H in G is an analytic subgroup of G.*

2. Analytic Tori. This section will discuss tori, which are certain subgroups and quotient groups of products of multiplicative groups. Since several different types of tori will be discussed, the word torus will never be used without an adjective to specify what kind of torus is meant.

Recall that the multiplicative group is given by

$$\mathbf{G}_m = \bigcup_{r>1} \mathcal{M}(\mathbf{C}_p\!<\!r^{-1}z, r^{-1}z^{-1}\!>).$$

The term **affine analytic torus,** or affine torus, will mean a product of multiplicative groups. An affine analytic n-torus is

$$T \cong \mathbf{G}_m \times \cdots \times \mathbf{G}_m,$$

where there are n terms in the product. Over a non-algebraically closed field this would be called a split affine analytic torus, but since we are working only with $\mathbf{C}_p$, the adjective split will be omitted.

An important subgroup of $\mathbf{G}_m$ is the affinoid analytic group

$$\mathbf{G}_{m,1} = \mathcal{M}(\mathbf{C}_p < z, z^{-1} >).$$

The term **affinoid n-torus** will mean

$$T_1 = \mathbf{G}_{m,1} \times \cdots \times \mathbf{G}_{m,1},$$

a product of n copies of $\mathbf{G}_{m,1}$. Again, since we are working over $\mathbf{C}_p$, the adjective split will be omitted. Note that the canonical reduction of $\mathbf{G}_{m,1}$ is the multiplicative group over $\mathbf{F}_p^a$, and that any affinoid torus T_1 reduces to an affine algebraic torus over $\mathbf{F}_p^a$.

We need the following topological theorem from [**Ber**] about the contractibility of certain subsets of $\mathbf{P}^n$.

Theorem 2.1 *([**Ber**] 6.1.5). The projective space $\mathbf{P}^n$ is contractible. Moreover, any subset of $\mathbf{P}^n$ whose complement lies in a union of closed analytic subsets different from $\mathbf{P}^n$ is contractible.*

Corollary 2.2. *The affine n-torus is contractible.*

Let T be an affine n-torus, and let Γ be a torsion free, discrete, subgroup of $T(\mathbf{C}_p)$. Then, Γ acts discretely and freely on T, so the quotient space $X_\Gamma = T/\Gamma$ is an analytic space. From the above corollary, T is simply connected. Hence, T is a universal cover for X_Γ, and therefore

$$\pi_1(X_\Gamma) \cong \Gamma.$$

If the rank of Γ is equal to n, then X_Γ is called a **complete analytic torus,** or a complete torus.

Just as in the case of complex analysis, X_Γ need not be the analytification of an algebraic variety. The complete torus X_Γ is the analytification of an Abelian variety if and only if the transcendence degree of the field of meromorphic functions on X_Γ is equal to n. However, unlike the complex analytic case, not all Abelian varieties are complete analytic tori either.

Note that if H is a subgroup of $T(\mathbf{C}_p)$, then the image of H in X_Γ is an abstract subgroup of $X_\Gamma(\mathbf{C}_p)$, but need not be closed in the analytic Zariski topology. However, Corollary 1.3 guarantees that the analytic Zariski closure of the image of H will be an analytic subgroup of X_Γ.

Remark. Note that as analytic spaces $\mathbf{A}^{1\times} \cong \mathbf{G}_m$. Therefore, Proposition 4 of Chapter I shows that there are no non-constant analytic maps from $\mathbf{A}^1$ into an affine analytic torus because there are no non-constant analytic maps from $\mathbf{A}^1$ into $\mathbf{A}^{1\times}$.

The next theorem in this section will be a theorem about maps from $\mathbf{A}^{1\times}$ into affine tori. I will use the notation $\mathbf{G}_m$ when I want to emphasize the group structure on $\mathbf{A}^{1\times}$. For example, an analytic map from $\mathbf{G}_m$ will usually be assumed to be a group homomorphism, but an analytic map from $\mathbf{A}^{1\times}$ into an analytic group will never be assumed to be a group homomorphism.

Theorem 2.3. *Let T be an affine analytic torus, and let $f\colon \mathbf{A}^{1\times} \to T$ be an analytic map. Then, f is an analytic group homomorphism $\phi\colon \mathbf{G}_m \to T$ composed with a translation $\tau\colon T \to T$.*

Proof: Since $T \cong \mathbf{G}_m \times \cdots \times \mathbf{G}_m$, the map f can be represented by an analytic map

$$f\colon \mathbf{A}^{1\times} \to \mathbf{G}_m \times \cdots \times \mathbf{G}_m,$$

given by

$$z \mapsto (f_1(z), \ldots, f_n(z)),$$

where

$$f_j\colon \mathbf{A}^{1\times} \to \mathbf{G}_m \hookrightarrow \mathbf{A}^1.$$

However, Proposition 4 of Chapter I implies that each function f_j is in fact algebraic, and therefore the map is given by

$$z \mapsto (c_1 z^{d_1}, \ldots, c_n z^{d_n}),$$

with $c_j \in \mathbf{C}_p$ and $d_j \in \mathbf{Z}$. Therefore, f is the composition of the group homomorphism

$$z \mapsto (z^{d_1}, \ldots, z^{d_n})$$

followed by the translation

$$(z_1, \ldots, z_n) \mapsto (c_1 z_1, \ldots, c_n z_n),$$

and hence the theorem.

Remark. This is markedly different from the complex analytic case, where the exponential function provides numerous maps from $\mathbf{C}^{\times}$ into affine tori which are not translations of group homomorphisms.

I will now derive some consequences of the above theorem for maps from $\mathbf{A}^{1\times}$ into complete analytic tori. First, I need to recall a lemma about characters on an abstract group, and then I need a lemma on the structure of connected analytic subgroups of tori.

Lemma 2.4. *Let G be an abstract group and let F be a field. The set of homomorphisms $G \to F^{\times}$ are linearly independent in the space of all F-valued functions on G.*

Proof: Let $\chi_1, \ldots, \chi_n: G \to F^{\times}$ be linearly dependent with $n > 1$ as small as possible. By the minimality of n, the linear relation can be written as

$$\sum_{i=1}^{n-1} a_i \chi_i + \chi_n = 0,$$

where each a_i is in $F^{\times}$. Since $\chi_1 \neq \chi_n$, there is a y in G such that $\chi_1(y) \neq \chi_n(y)$. The following two equations

$$\sum_{i=1}^{n-1} a_i \chi_i(x)\chi_i(y) + \chi_n(x)\chi_n(y) = \sum_{i=1}^{n-1} a_i \chi_i(xy) + \chi_n(xy) = 0$$

and

$$\sum_{i=1}^{n-1} a_i \chi_i(x)\chi_n(y) + \chi_n(x)\chi_n(y) = 0$$

are satisfied for all x in G. Subtracting one equation from the other leaves

$$\sum_{i=1}^{n-1} a_i(\chi_i(y) - \chi_n(y))\chi_i = 0,$$

contradicting the minimality of n.

Lemma 2.5. *Let T be an affine (resp. affinoid, resp. complete) torus, and let G be a connected analytic subgroup of T. Then, G is again an affine (resp. affinoid, resp. complete) torus.*

Proof: First assume that T is affine (resp. affinoid). Let $\chi_1, \ldots, \chi_n$ be analytic homomorphisms from T to $\mathbf{G}_m$ (resp. $\mathbf{G}_{m,1}$) such that

$$\Psi = (\chi_1, \ldots, \chi_n): T \to \mathbf{G}_m \times \cdots \times \mathbf{G}_m \quad (\text{resp. } \mathbf{G}_{m,1} \times \cdots \times \mathbf{G}_{m,1})$$

is an isomorphism. Let

$$T' = \Psi^{-1}(\mathbf{G}_{m,1} \times \cdots \times \mathbf{G}_{m,1}) \quad (\text{resp. } T' = T),$$

and let $G' = G \cap T'$.

Let M be the free $\mathbf{Z}$-module generated by $\chi_1, \ldots, \chi_n$. Let N be the submodule of M defined by $\chi \in N$ if $\chi(g) = 1$ for every g in $G(\mathbf{C}_p)$. Every χ in M is a proper analytic group homomorphism from G into $\mathbf{G}_m$ (resp. $\mathbf{G}_{m,1}$). The Proper Mapping Theorem (see **[BGR]** 9.6.3/3) implies that $\chi(G)$ is an analytic subgroup of $\mathbf{G}_m$ (resp. $\mathbf{G}_{m,1}$). Since G is connected and the only connected analytic subgroups of $\mathbf{G}_m$ (resp. $\mathbf{G}_{m,1}$) are the indentity subgroup and all of $\mathbf{G}_m$ (resp. $\mathbf{G}_{m,1}$), χ is either trivial on G or maps G surjectively to $\mathbf{G}_m$ (resp. $\mathbf{G}_{m,1}$). Therefore, M/N is a torsion free, finitely generated, $\mathbf{Z}$-module, hence free. So, let $\rho_1, \ldots, \rho_r \in M$ be a basis for M/N.

Define

$$\Theta: G \to \mathbf{G}_m \times \cdots \times \mathbf{G}_m \quad (\text{resp. } \mathbf{G}_{m,1} \times \cdots \times \mathbf{G}_{m,1}),$$

where the product is taken r times, by sending

$$g \mapsto (\rho_1(g), \ldots, \rho_r(g)).$$

Because, $\rho_1, \ldots, \rho_r$ span M/N, the map Θ is injective. To complete the proof, we need to show that Θ is surjective.

Since Θ is a proper analytic map, the Proper Mapping Theorem again implies that $\Theta(G)$ is an analytic subvariety. So, to show that Θ is surjective, it suffices to show that the image of Θ contains an open set. To do this, we will show

$$\Theta(G') = \mathbf{G}_{m,1} \times \cdots \times \mathbf{G}_{m,1}.$$

Indeed, since G' is affinoid, we can write $G' = \mathcal{M}(A)$. Then

$$\Theta: G' \to \mathbf{G}_{m,1} \times \cdots \times \mathbf{G}_{m,1}$$

comes from the map

$$\mathbf{C}_p < t_1, t_1^{-1}, \ldots, t_r, t_r^{-1} > \longrightarrow A$$

given by $t_j \mapsto \rho_j$. (Note that $|\rho_j|_{\text{sup}} = 1$.) We need to show that this latter map is injective. So, assume that it is not injective. That means there exist a_ν in $\mathbf{C}_p$ such that

$$\sum_{\nu \in \mathbf{Z}^r} a_\nu \rho_1^{\nu_1} \cdots \rho_r^{\nu_r} = 0$$

with $\sup_\nu |a_\nu|_p = 1$. But, this implies

$$\sum_\nu \tilde{a}_\nu \tilde{\rho}_1^{\nu_1} \cdots \tilde{\rho}_r^{\nu_r} = 0$$

in $\widetilde{A}$. But, this last relation has only finitely many non-zero terms. The previous lemma therefore implies that not all of the above monomials are distinct. In other words, there exist integers $e_1, \ldots e_r$, not all zero, such that

$$\tilde{\rho}_1^{e_1} \cdots \tilde{\rho}_r^{e_r} = 1.$$

This implies that

$$|\rho_1^{e_1} \cdots \rho_r^{e_r} - 1|_{\text{sup}} < 1.$$

If $\rho_1^{e_1}\cdots\rho_r^{e_r}$ were not equal to one, then the fact that G' is connected implies that

$$\rho_1^{e_1}\cdots\rho_r^{e_r}\colon G'\to \mathbf{G}_{m,1}$$

is surjective, and this contradicts

$$|\rho_1^{e_1}\cdots\rho_r^{e_r}-1|_{\text{sup}}<1.$$

Therefore, $\rho_1^{e_1}\cdots\rho_r^{e_r}\equiv 1$ on G'. However, since G' contains an open subset of G and since $\rho_1^{e_1}\cdots\rho_r^{e_r}$ is analytic on G, $\rho_1^{e_1}\cdots\rho_r^{e_r}$ is identically one on all of G. This contradicts the linear independence of $\rho_1,\ldots\rho_r$, so Θ must have been surjective. This completes the proof in the affine and affinoid cases.

If T is a complete torus, then $T=T'/\Gamma$, where T' is an affine torus. Let G' be the connected component of the identity in the inverse image of G in T'. Then, G' is a connected analytic subgroup of T', so will again be an affine torus by the above. The group $G'\cap\Gamma$ will be a discrete, torsion free, subgroup of G' and $G'/G'\cap\Gamma=G$, so G is a complete torus. This completes the proof of Lemma 2.5.

The following corollaries are then immediate consequences of Theorem 2.3 and Lemma 2.5.

Corollary 2.6. *The analytic Zariski closure of the image of an analytic map $f\colon \mathbf{A}^{1\times}\to X$ into a complete analytic torus X is the translate of a complete analytic subtorus.*

Corollary 2.7. *Let X be a complete analytic torus. Let Y be an analytic subvariety of X that does not contain any translates of positive dimensional complete analytic subtori in X. Then, if $f\colon \mathbf{A}^{1\times}\to Y$ is an analytic map, it must be constant. In other words, Y is p-adic Brody hyperbolic.*

3. Reduction and Uniformization of Abelian Varieties. In this section the analytification of Abelian varieties over $\mathbf{C}_p$ will be discussed. As was noted previously, not all Abelian varieties are complete tori, so one can expect that the uniformization of Abelian varieties over $\mathbf{C}_p$ is more complicated than it is over $\mathbf{C}$. The main theorems in this section are the Semi-Abelian Reduction Theorem and a uniformization theorem. First though, some more discussion of reductions is necessary.

A **formal analytic space** X is an analytic space together with a fixed equivalence class of formal coverings. (The definitions of a formal covering and what it means for two formal coverings to be equivalent were given in Section 4 of Chapter IV.) Such a space is denoted by $(X,\mathfrak{U})$, where $\mathfrak{U}$ is a formal covering of X representing its equivalence class. A morphism

$$\phi\colon (X,\mathfrak{U})\to(Y,\mathfrak{V})$$

of formal analytic spaces is a morphism $\phi: X \to Y$ of analytic spaces such that there exists a formal covering $\mathcal{U}'$ of X equivalent to $\mathcal{U}$ and a formal covering $\mathcal{V}'$ of Y equivalent to $\mathcal{V}$ such that for every $U \in \mathcal{U}'$, there exists a $V \in \mathcal{V}'$ such that $\phi(U) \subset V$. Recall that such a morphism induces a morphism

$$\tilde{\phi}: \widetilde{X_{\mathcal{U}}} \to \widetilde{Y_{\mathcal{V}}},$$

and this is the whole point of considering the category of formal analytic spaces.

One easily sees that products exist in the category of formal analytic spaces, so it makes sense to talk about **formal analytic groups**, which are group objects in the category of formal analytic spaces.

There is an obvious functor from the category of formal analytic spaces to the category of analytic spaces, and from the category of formal analytic groups to the category of analytic groups, namely the functor which simply forgets the formal cover. By a theorem of Bosch [**Bos**], this functor from the category of formal analytic groups to the category of analytic groups is fully faithful, so in particular *an analytic group can have at most one formal analytic group structure.* Of course it might be that some analytic groups cannot be made into formal analytic groups at all, and in fact we will see that this is the case. Note that the problem is not in finding a formal covering for an analytic group G, but rather that not all analytic groups G have formal coverings such that the multiplication map from the product of G with itself will then be a morphism in the category of formal analytic spaces. This last property is equivalent to the existence of a formal covering $\mathcal{U}$ such that $\widetilde{G_{\mathcal{U}}}$ is a group variety.

The following theorem tells us that compact formal analytic groups are topologically trivial. The idea is that they contract onto the inverse image of the generic point of the reduction.

Theorem 3.1 *(*[**Ber**] *6.4.2). A connected compact reduced formal analytic group G is contractible.*

Corollary 3.2. *If X is a complete analytic torus, then it does not have a formal analytic group structure.*

Proof: Complete analytic tori have non-trivial fundamental groups.

Remark. This illustrates very well the general philosophy of what can happen when one reduces mod p. If G is an analytic group with a formal analytic group structure $\mathcal{U}$, then $\widetilde{G_{\mathcal{U}}}$ is a group variety over $\mathbf{F}_p^a$. Therefore, analytic subgroups of analytic groups which can be reduced to group varieties do not contribute to the topology of G. Therefore, if G has non-trivial topology, then it is because things degenerate sufficiently mod p so that one does not get a group variety.

A formal analytic group $(G, \mathfrak{U})$ is said to have **Abelian reduction** if $\widetilde{G_{\mathfrak{U}}}$ is an Abelian variety, and $(G, \mathfrak{U})$ is said to have **semi-Abelian reduction** if $\widetilde{G_{\mathfrak{U}}}$ is a semi-Abelian variety. A **semi-Abelian variety** is an extension of an Abelian variety by a torus. In view of Bosch's result, an analytic group G will be said to have Abelian or semi-Abelian reduction if G can be given the structure of a formal analytic group with Abelian or semi-Abelian reduction.

Theorem 3.3 (Semi-Abelian Reduction). *Let A be the analytification of an Abelian variety. Then, there exists a compact analytic subgroup N in A such that*

(a) N is a connected formal analytic group with semi-Abelian reduction.
(b) There is a formal analytic group B with Abelian reduction, which is the analytification of an Abelian variety, and there is an affinoid torus T_1 in N together with an exact sequence

$$1 \to T_1 \to N \to B \to 1.$$

Furthermore, the group N is uniquely determined by condition (a). The affinoid torus T_1 is also unique. Note that an affinoid torus is a formal analytic group, so the exact sequence above reduces to the exact sequence defining $\widetilde{N}$ as a semi-Abelian variety.

Proof: See [**Ber**] Section 6.5 and [**BL2**] Theorem 8.2.

Theorem 3.4 (Uniformization Theorem). *Let A be the analytification of an Abelian variety. Let T_1, N and B be as in Theorem 3.3. Let T be an affine analytic torus with the same rank as T_1, and embed T_1 into T by*

$$T_1 \cong \mathbf{G}_{m,1} \times \cdots \times \mathbf{G}_{m,1} \hookrightarrow \mathbf{G}_m \times \cdots \times \mathbf{G}_m \cong T.$$

Then,

(a) $G = T \times N/\text{diagonal}$ exists as an analytic quotient, and there is an exact sequence

$$1 \to T \to G \to B \to 1,$$

so G is a semi-Abelian variety. Here "diagonal" refers to the image of T_1 along the diagonal in $T \times N$.
(b) The immersion $N \hookrightarrow A$ extends uniquely to a surjective analytic group homomorphism

$$\phi: G \to A,$$

which is also a topological covering map.
(c) $\Gamma = \ker \phi$ is a discrete subgroup in $G(\mathbf{C}_p)$, which is free and whose rank is equal to the rank of T_1.

(d) G is simply connected and $\pi_1(A) \cong \Gamma$.

Proof: See [**Ber**] Section 6.5 and [**BL2**] Theorem 8.8.

4. Bloch's Conjecture. In this section, analytic maps from $\mathbf{A}^1$ and $\mathbf{A}^{1\times}$ into subvarieties of Abelian varieties will be studied, and a proof will be given for a p-adic analogue of Bloch's conjecture. This section starts with

Theorem 4.1. *Let B be the analytification of an Abelian variety. Assume that B can be provided with a formal analytic group structure and that as a formal analytic group, B has Abelian reduction. If f is an analytic map from $\mathbf{A}^1$ or $\mathbf{A}^{1\times}$ into B, then f must be constant.*

Proof: Let $\pi: B \to \widetilde{B}$ denote the reduction of B, so $\widetilde{B}$ is an Abelian variety. Let x be a point in B, and assume that $\tilde{x}$ in $\widetilde{B}$ is not a closed point. Then, x is not in the image of f. Indeed, if x were in the image of f, then there would be a point z in $\mathbf{A}^1$ or $\mathbf{A}^{1\times}$ with $f(z) = x$, and that would imply the existence of a non-zero homomorphism from $\mathcal{K}(x)$ into $\mathcal{K}(z)$. This would then imply

$$\mathcal{K}(\tilde{x}) \hookrightarrow \widetilde{\mathcal{K}(x)} \hookrightarrow \widetilde{\mathcal{K}(z)},$$

by Proposition IV.4.5. But, this is impossible because by Proposition IV.1.3, $\widetilde{\mathcal{K}(z)}$ is either $\mathbf{F}_p^a$ or the field of rational functions in one variable over $\mathbf{F}_p^a$, and if there were a non-zero map from $\mathcal{K}(\tilde{x})$ into $\widetilde{\mathcal{K}(z)}$, then there would be a non-constant rational image of a $\mathbf{P}^1$ inside $\widetilde{B}$ since $\tilde{x}$ is not closed. This would contradict the fact that $\widetilde{B}$ is an Abelian variety, so x cannot be in the image of f.

This implies that the image of f lies entirely above the closed points of $\widetilde{B}$. But, $\mathbf{A}^1$ and $\mathbf{A}^{1\times}$ are arc-connected, so $f(\mathbf{A}^1)$ or $f(\mathbf{A}^{1\times})$ must also be arc-connected. Therefore, if the image of f contained points above more than one closed point in $\widetilde{B}$, Proposition IV.4.6 would imply that the image of f also contained at least one point lying above a non-closed point in $\widetilde{B}$, which, by the above, cannot be the case. Therefore, the image of f lies above a single closed point $\tilde{x}$, which is smooth since $\widetilde{B}$ is an Abelian variety. Thus, Proposition IV.4.6 implies

$$\pi^{-1}(\tilde{x}) \cong \mathring{\mathbf{B}}^n,$$

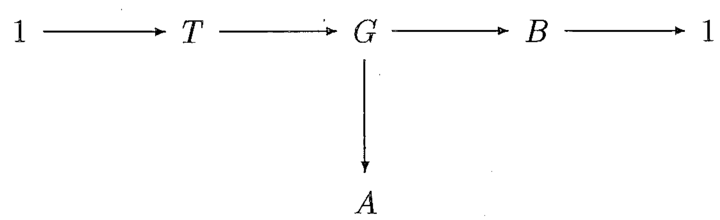
and f can be considered as a map from $\mathbf{C}_p$ or $\mathbf{C}_p^\times$ into $\mathring{\mathbf{B}}^n$. Proposition 4 of Chapter I then implies that f is constant since $z_j \circ f$ is bounded and hence constant for each coordinate function z_j on $\mathring{\mathbf{B}}^n$.

Now I state two theorems about analytic maps into Abelian varieties, whose proofs are similar.

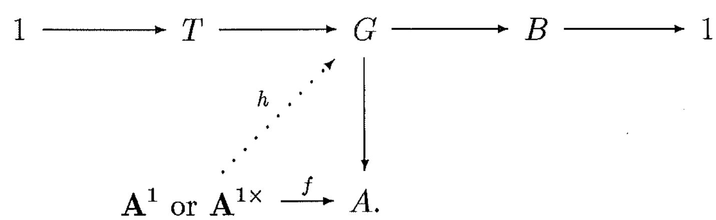
Theorem 4.2 *If $f: \mathbf{A}^1 \to A$ is an analytic map into an Abelian variety, then f must be constant.*

Theorem 4.3. *Let $X \subset A$ be a subvariety of an Abelian variety. If $f\colon \mathbf{A}^{1\times} \to X$ is an analytic map, then the image of f is contained in the translate of an Abelian subvariety E of A contained in X. Furthermore, E does not have Abelian reduction if f is non-constant.*

Proof of Theorems 4.2 and 4.3. By Theorem 3.4, there exists an Abelian variety B with Abelian reduction, an affine analytic torus T, and an analytic group G such that the diagram



commutes, and the vertical arrow $G \to A$ is a universal covering map for A. This implies that the map $f\colon \mathbf{A}^1$ or $\mathbf{A}^{1\times} \to A$ lifts to a map $h\colon \mathbf{A}^1$ or $\mathbf{A}^{1\times} \to G$ as follows



The image of either $\mathbf{A}^1$ or $\mathbf{A}^{1\times}$ in B is a point by Theorem 4.1. Therefore, the image of h in G is contained in a translate of T.

For Theorem 4.2 all that remains to say is that, as noted in the remark preceding Theorem 2.3, the map h must then be constant.

We now complete the proof of Theorem 4.3. Theorem 2.3 says that all analytic maps from $\mathbf{A}^{1\times}$ into T are translates of group homomorphisms, so the image of h in G is also the translate of a group homomorphism. Thus, the image of f in A is again the translate of a group homomorphism. Now by Lemma 1.2, the analytic Zariski closure of the image of f in A is the translate of an Abelian subvariety E in A. However, X is closed in the Zariski topology, so this translate is contained in X.

Finally, Theorem 4.1 implies that E does not have Abelian reduction if f is not constant.

Corollary 4.4. *If X is a subvariety of an Abelian variety such that every translated positive dimensional Abelian subvariety contained in X has Abelian reduction, then X is p-adic Brody hyperbolic. In particular, if X does not contain the translate of a positive dimensional Abelian subvariety, then X is p-adic Brody hyperbolic.*

Theorem 4.5 (Bloch's Conjecture). *Let X be a smooth projective variety with ample irregularity, which by definition means that* $\dim H^1(X, \mathcal{O}_X)$ *is larger than* $\dim X$. *Then, every analytic map $f: \mathbf{A}^{1\times} \to X$ is contained in a proper algebraic subvariety.*

Remark. Theorems 4.5 and 4.3 can be used to give an alternate proof for Theorem V.4.1 because every smooth projective curve X of postive genus can be embedded into an Abelian variety.

Proof: Let A be the Albanese variety of X and Y the image of X in A. Then, by assumption, Y is a proper subvariety of A, and Y cannot be the translate of a proper Abelian subvariety since $Y(\mathbf{C}_p)$ must generate $A(\mathbf{C}_p)$ as a group. However, the image of $\mathbf{A}^{1\times}$ in Y is the translate of a proper Abelian subvariety of A contained in Y by Theorem 4.3. Therefore, the image of f is contained in a proper subvariety of Y, and hence the image of $\mathbf{A}^{1\times}$ under f in X is also contained in a proper subvariety since the map from X to A is algebraic.

In light of this result, I make the following conjectures:

Conjecture 4.6. *Let X be a projective variety defined over $\mathbf{C}_p$. If X is of general type, then the image of any analytic map $f: \mathbf{C}_p^\times \to X$ is contained in a proper subvariety of X.*

Conjecture 4.7. *Let X be a projective variety defined over $\mathbf{C}_p$. If every subvariety of X (including X itself) is of general type, then X is p-adic Brody hyperbolic. Furthermore, if X is p-adic Brody hyperbolic, then either every subvariety of X (including X itself) is of general type, or there exists an Abelian variety A with Abelian reduction and a non-constant algebraic morphism $\phi: A \to X$.*

CHAPTER VII

An Analogue of the Kobayashi Semi-Distance

Over the complex numbers, there are several different notions of hyperbolicity. So far, I have been discussing a p-adic analogue to what is known as "Brody hyperbolicity" – that is, the non-existence of non-constant holomorphic maps $\mathbf{C} \to X$. There is also a metric notion of hyperbolicity called "Kobayashi hyperbolicity." Brody's Theorem [**Br**] says that for compact complex analytic spaces, these two notions are equivalent. However, there is an example to show that they are not, in general, equivalent for non-compact manifolds. (See Example 3 on page 79 of [**L1**].) The notion of Kobayashi hyperbolic is sometimes more useful when studying complex subspaces of larger complex spaces. This is largely because it gives rise to the notion of a complex space being "hyperbolically embedded" in a larger space. See Chapter II of [**L1**] for the precise definitions here.

In this chapter, I suggest a p-adic notion of hyperbolicity analogous to the notion of Kobayashi hyperbolicity over $\mathbf{C}$. In his proof that Kobayashi hyperbolicity and Brody hyperbolicity are equivalent for compact, complex, analytic spaces, Brody makes essential use of compactness, and the fact that $\mathbf{C}_p$ is not locally compact prevents Brody's proof from being directly translated to the p-adic case. In Section 3, I look at some examples that suggest an analogue of Brody's Theorem might be true p-adically. At the same time, I also give some examples illustrating the relationship between hyperbolicity and the existence (or non-existence) of rational curves. I complete this chapter by mentioning some open problems in these directions.

Notational Remark. By analytic spaces X in this chapter, I will mean analytic spaces in the sense of Berkovich. However, as I have previously remarked, such analytic spaces are not metrizable, and the main point of this chapter is to discuss a "metric." Therefore, I will most often be dealing only with the $\mathbf{C}_p$ points of the space, which I will continue to denote $X(\mathbf{C}_p)$.

1. Distance Decreasing. As a manifold, the unit disc in the complex plane can be provided with a number of Riemannian metrics. One of the most useful metrics is the "**hyperbolic**" or "**Poincaré**" metric given by

$$ds^2 = \frac{|dz|^2}{\left(1 - |z|^2\right)^2}.$$

This metric is sometimes defined to be a constant multiple involving 2's and π's times the above expression. It is called hyperbolic because it has constant negative Gaussian curvature ($= -1$ if the normalizing constant is chosen properly). One of the reasons that this metric is so useful is that the Schwarz-Pick Lemma says that if f is a holomorphic map from the unit disc to itself, then it is distance decreasing in the hyperbolic metric, and if f is biholomorphic, then it is an isometry in the hyperbolic metric. Many results in hyperbolic geometry follow from this distance decreasing property. In this section, I show that p-adic analytic maps from the unit ball $\mathbf{B}$ to itself are distance decreasing in the standard p-adic norm on $\mathbf{B}(\mathbf{C}_p)$. Therefore, we can develop a p-adic analogue to the complex analytic situation by taking the unit ball together with its standard norm as our model.

Remark. In a recent preprint, Lin Weng [**We**] suggests an alternate definition for a hyperbolic distance on the "open" unit ball $\mathring{\mathbf{B}}(L)$ by defining

$$d_{\text{hyp}}(a,b) = \log_q \frac{1+|b-a|_p}{1-|b-a|_p},$$

where L is a *finite* extension of $\mathbf{Q}_p$, q is the cardinality of the residue class field of L, and $\log_q$ means the logarithm with base q. This definition has the advantage of "looking more like" the complex analytic distance. I could modify this definition to work over $\mathbf{C}_p$ by taking the log with base p, but everything I will prove in this chapter will be true no matter which of the two models I start with. Since I do not at the moment see a clear reason to work with one or the other, I prefer to work with the "closed" unit ball $\mathbf{B}$ and the standard norm, rather than the "open" unit ball and the logarithmic norm.

Proposition 1.1. *Let $f\colon \mathbf{B}^n \to \mathbf{B}^n$ be an analytic map, and let $\tilde{f}\colon \mathbf{A}^n_{\mathbf{F}_p^a} \to \mathbf{A}^n_{\mathbf{F}_p^a}$ denote the reduction of f, then f is an isomorphism if and only if $\tilde{f}$ is an isomorphism.*

Proof: See [**BGR**] 5.1.3/5.

Corollary 1.2. *Let $f\colon \mathbf{B} \to \mathbf{B}$ be an analytic map given by*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Then, f is an isomorphism if and only if

$$|a_0|_p \leq 1, \; |a_1|_p = 1, \; \textit{and } |a_k|_p < 1, \; \textit{for all } k \geq 2.$$

Proof: The reduced map $\tilde{f}$, which is a polynomial in one variable, will be an isomorphism if and only if it has degree one.

Proposition 1.3. *If $f\colon \mathbf{B} \to \mathbf{B}$ is an analytic map, then for all $z, w \in \mathbf{B}(\mathbf{C}_p)$,*

$$|f(z) - f(w)|_p \leq |z-w|_p.$$

Furthermore, if f is an isomorphism, then

$$|f(z) - f(w)|_p = |z - w|_p$$

for all z, w in $\mathbf{B}(\mathbf{C}_p)$.

Proof: Let $f(z)$ be given by $\sum a_k z^k$. Now, because f maps into $\mathbf{B}$, one has $|a_k|_p \leq 1$ for all k. Therefore,

$$\begin{aligned} f(z) - f(w) &= \sum_{k=1}^{\infty} a_k (z^k - w^k) \\ &= \sum_{k=1}^{\infty} a_k (z - w)(z^{k-1} + wz^{k-2} + \cdots + w^{k-2}z + w^{k-1}) \\ &= (z - w) \sum_{k=1}^{\infty} a_k (z^{k-1} + wz^{k-2} + \cdots + w^{k-2}z + w^{k-1}). \end{aligned}$$

Since $|a_k|_p \leq 1, |z|_p \leq 1$, and $|w|_p \leq 1$, everything inside the last sum has norm ≤ 1. Hence,

$$|f(z) - f(w)|_p \leq |z - w|_p.$$

If, in addition, f is an isomorphism, then $|a_k|_p < 1$ for $k \geq 2$, and $|a_1|_p = 1$ by Corollary 1.2. This then implies

$$|f(z) - f(w)|_p = |a_1|_p |z - w|_p = |z - w|_p.$$

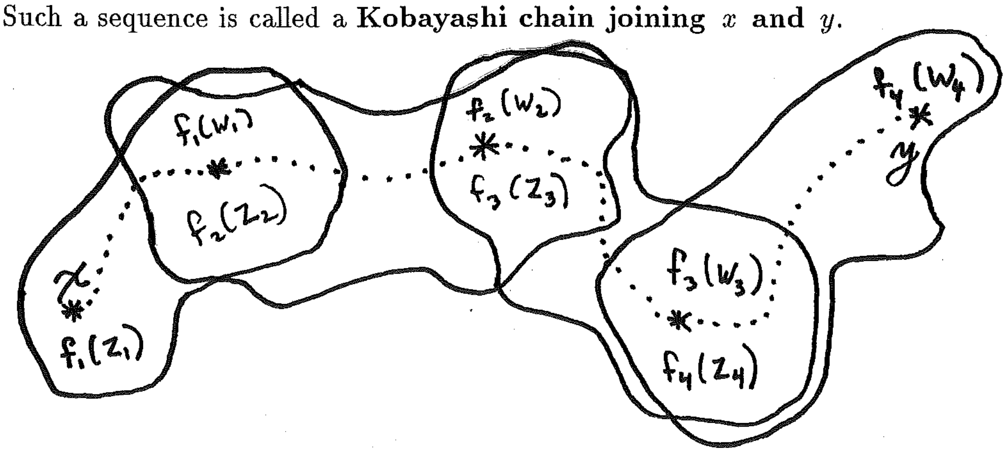
2. Kobayashi Semi-Distance. Let X be a p-adic analytic space, and let $x, y \in X(\mathbf{C}_p)$. Suppose that there is a sequence of analytic maps

$$f_j : \mathbf{B} \to X, \qquad j = 1, \ldots, m,$$

and points $z_j, w_j \in \mathbf{B}(\mathbf{C}_p)$ such that

$$f_1(z_1) = x, \quad f_m(w_m) = y, \quad \text{and} \quad f_j(w_j) = f_{j+1}(z_{j+1}), \quad j = 1, \ldots, m-1.$$

Such a sequence is called a **Kobayashi chain joining x and y.**



The Kobayashi semi-distance is then defined by

$$d(x,y) = \inf \sum_{j=1}^{m} |z_j - w_j|_p,$$

where the infimum is taken over all Kobayashi chains (f_i, z_i, w_i) joining x and y. If there are no Kobayashi chains joining x and y, then define $d(x,y) = \infty$. The Kobayashi semi-distance is the only (semi-) distance used in this chapter, so henceforth d will always refer to the Kobayashi semi-distance.

It is clear that d is symmetric in x and y, and that d satisfies the triangle inequality, though not necessarily the stronger non-Archimedean triangle inequality. It is, however, possible that $d(x,y) = 0$ but $x \neq y$, as we will see in the second example below.

Remark 2.1. In the case of connected complex analytic spaces, the Kobayashi semi-distance is always finite, but this is definitely not the case for p-adic analytic spaces. For example, let X be a smooth projective curve of genus ≥ 1 with good reduction. Then, as in the proof of Theorem V.4.1, one sees that the image of any analytic map $f\colon \mathbf{B} \to X$ must lie over a single closed point $\tilde{x}$ in the reduction $\widetilde{X}$. Thus, if $x, y \in X(\mathbf{C}_p)$ such that $\tilde{x} \neq \tilde{y}$, then $d(x,y) = \infty$.

Example 2.2. Let $X = \mathbf{B}$. Since $|f(z) - f(w)|_p \leq |z - w|_p$ for all $z, w \in \mathbf{B}(\mathbf{C}_p)$ and all analytic maps f by Proposition 1.3, we see that $d(z,w) = |z - w|_p$ for all $z, w \in \mathbf{B}(\mathbf{C}_p)$. Hence, the Kobayashi semi-distance on $\mathbf{B}$ coincides with the distance induced from the standard norm on $\mathbf{B}(\mathbf{C}_p)$.

Example 2.3. Let $X = \mathbf{A}^1$, and let $x, y \in \mathbf{A}^1(\mathbf{C}_p) = \mathbf{C}_p$. Let w be a non-zero element of $\mathbf{B}(\mathbf{C}_p)$, and define

$$f\colon \mathbf{B} \to \mathbf{A}^1, \quad \text{by} \quad f(z) = x + \left(\frac{y-x}{w}\right) z.$$

Then, $d(x,y) \leq |w|_p$. But, the choice of w was arbitrary, so w can be chosen with $|w|_p$ arbitrarily small, and hence

$$d(x,y) = 0 \text{ for every } x, y \in \mathbf{A}^1(\mathbf{C}_p).$$

Notice that in this example we have used the fact that we can make arbitrarily large dilations of the disc when we map into $\mathbf{A}^1$.

Proposition 2.4. *Let X and X' be p-adic analytic spaces, and let $f\colon X \to X'$ be an analytic map. Let d and d' denote the Kobayashi semi-distances on X and X' respectively. Then,*

$$d'(f(x), f(y)) \leq d(x,y)$$

for all $x, y \in X(\mathbf{C}_p)$. In other words, analytic maps are distance decreasing in the Kobayashi semi-distance. If in addition, f is an analytic isomorphism, then f is an isometry for the Kobayashi semi-distance.

Proof: Composition with f makes any analytic map $g: \mathbf{B} \to X$ into a map from $\mathbf{B}$ into X'. Therefore, composition by f makes any Kobayashi chain connecting x and y in X into a Kobayashi chain of the same length connecting $f(x)$ and $f(y)$ in X'. The statement about isomorphisms then follows by symmetry.

Corollary 2.5. *Let X be an analytic space and $f: \mathbf{A}^1 \to X$ an analytic map. If $x, y \in X(\mathbf{C}_p)$ are two points in the image of f, then $d(x, y) = 0$. Therefore, if X is an analytic space admitting a non-constant analytic map $f: \mathbf{A}^1 \to X$, then the Kobayashi semi-distance on X fails to be a distance.*

The next proposition relates the Kobayashi semi-distance on a product space to the Kobayashi semi-distance on each of the factor spaces.

Proposition 2.6. *Let X and Y be p-adic analytic spaces, and let $X \times Y$ denote the product space. Let d_X, d_Y and $d_{X\times Y}$ denote the Kobayashi semi-distances on X, Y and $X \times Y$ respectively. For all $x, x' \in X(\mathbf{C}_p)$ and all $y, y' \in Y(\mathbf{C}_p)$, one has*

$$d_X(x, x') + d_Y(y, y') \geq d_{X\times Y}((x, y), (x', y')) \geq \max\{d_X(x, x'), d_Y(y, y')\}.$$

Proof: The second inequality follows from the fact that the projection maps $X \times Y \to X$ and $X \times Y \to Y$ are analytic, hence distance decreasing for the Kobayashi semi-distance by Proposition 2.4. The first inequality is trivial if either term on the left-hand side is infinite, so assume that both $d_X(x, x')$ and $d_Y(y, y')$ are finite. Map $X \to X \times Y$ by $\cdot \mapsto (\cdot, y)$ and $Y \to X \times Y$ by $\cdot \mapsto (x', \cdot)$, so that any Kobayashi chain connecting x to x' in X will connect (x, y) to (x', y) in $X \times Y$, and any Kobayashi chain connecting y to y' in Y will connect (x', y) and (x', y') in $X \times Y$. Then,

$$\begin{aligned} d_{X\times Y}((x, y), (x', y')) &\leq d_{X\times Y}((x, y), (x', y)) + d_{X\times Y}((x', y), (x', y')) \\ &\leq d_X(x, x') + d_Y(y, y'), \end{aligned}$$

by the triangle inequality.

Example 2.7. If $X = \mathbf{B}^n = \mathbf{B} \times \cdots \times \mathbf{B}$, then

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{1\leq j\leq n} |y_j - x_j|_p.$$

To see this, it suffices to assume that

$$x_1 = \cdots = x_n = 0 \quad \text{and} \quad |y_n|_p \geq |y_{n-1}|_p \geq \cdots \geq |y_2|_p \geq |y_1|_p$$

because isomorphisms preserve the Kobayashi semi-distance. Of course, we may also assume that $y_n \neq 0$. Now, let

$$f: \mathbf{B}^n \to \mathbf{B}^n \quad \text{be given by} \quad z \mapsto \left(\frac{y_1}{y_n} z, \frac{y_2}{y_n} z, \ldots, \frac{y_{n-1}}{y_n} z, z \right).$$

Since f is distance decreasing,

$$d((0, \ldots, 0), (y_1, \ldots, y_n)) = d(f(0), f(y_n)) \leq |y_n|_p.$$

But, the second inequality in Proposition 2.6 tells us that

$$|y_n|_p = \max_{1 \leq j \leq n} |y_j|_p \leq d((0, \ldots, 0), (y_1, \ldots, y_n)).$$

Therefore, $d((0, \ldots, 0), (y_1, \ldots, y_n)) = |y_n|_p$ as desired.

Proposition 2.8. *Let X be a non-singular p-adic analytic space. Then, the Kobayashi semi-distance d is a continuous function on $X(\mathbf{C}_p) \times X(\mathbf{C}_p)$.*

Proof: As usual, by the triangle inequality, it suffices to check that if $x_m \to x$ in $X(\mathbf{C}_p)$, then $d(x_m, x) \to 0$. By the definition of non-singular, there is an analytic map $f: \mathbf{B}^n \to X$ such that $f(0) = x$ and such that f is an isomorphism onto its range. Hence, for m sufficiently large, let $z_m \in \mathbf{B}^n$ be the point such that $f(z_m) = x_m$. Then, the sequence z_m tends to zero, and hence $d(z_m, 0) \to 0$ because the Kobayashi distance on the n-ball is just the standard distance by Example 2.7. Because analytic maps are distance decreasing in the Kobayashi semi-distance,

$$d(x_m, x) = d(f(z_m), f(0)) \leq d(z_m, 0) \to 0,$$

so $d(x_m, x) \to 0$ as was to be shown.

We will see that whether or not the Kobayahsi semi-distance on a p-adic analytic space X is a distance (i.e. $d(x, y) = 0$ implies $x = y$) is closely related to the existence of non-constant analytic maps $\mathbf{A}^1 \to X$. We have already seen that if X is an algebraic variety defined over a number field, then the existence of a non-constant p-adic analytic map from $\mathbf{A}^1$ into X is a rarer phenomenon than is the existence of non-constant complex analytic maps from $\mathbf{C}$ into the variety. In particular, there are non-constant complex analytic maps from $\mathbf{C}$ into Abelian varieties, but no non-constant p-adic analytic maps from $\mathbf{A}^1$ into Abelian varieties. Because hyperbolicity (over $\mathbf{C}$) is conjecturally equivalent to every subvariety being of general type (pseudo-canonical), I do not want to call the non-existence of non-constant p-adic analytic maps $f: \mathbf{A}^1 \to X$ "hyperbolic." Instead, I will use the term "anti-hyperbolic" to mean the opposite, just as varieties such that the negative of the canonical class is ample are sometimes called "anti-canonical."

Definition 2.9. A p-adic analytic space X is called **Brody anti-hyperbolic** if there exists a non-constant p-adic analytic map $f\colon \mathbf{A}^1 \to X$.

In this terminology, Theorem V.4.1 says that if X is a Brody anti-hyperbolic non-singular projective curve, then $X \cong \mathbf{P}^1$.

Definition 2.10. A p-adic analytic space X is called **Kobayashi anti-hyperbolic** if there exist two distinct points $x, y \in X(\mathbf{C}_p)$ such that $d(x,y) = 0$, where d is the Kobayashi semi-distance.

We have seen that $\mathbf{A}^1$ is Kobayashi anti-hyperbolic and that $\mathbf{B}$ and $\mathbf{B}^n$ are not Kobayashi anti-hyperbolic. With this new terminology, Corollary 2.5 becomes

If X is a Brody anti-hyperbolic p-adic analytic space, then X is also Kobayashi anti-hyperbolic.

In the next section, I examine some special cases to support the idea that, at least for projective algebraic varieties, the converse of the above statement might also be true. First though, I prove some useful lemmas.

Lemma 2.11. *Let $\pi\colon \widehat{X} \to X$ be an analytic map which is a finite covering map on the underlying topological spaces. If $\widehat{X}$ is not Kobayashi anti-hyperbolic, then X is not Kobayashi anti-hyperbolic either.*

Proof: Let d and $\hat{d}$ denote the Kobayashi semi-distances on X and $\widehat{X}$ respectively. Let $x, y \in X(\mathbf{C}_p)$ be two points such that $d(x,y) = 0$. Fix a lift $\hat{x}$ of x. Because $\mathbf{B}$ is contractible and π is a covering map, any Kobayashi chain joining x to y in X will lift to a Kobayashi chain of the same length joining $\hat{x}$ to some lift $\hat{y}$ of y in $\widehat{X}$. Because $d(x,y) = 0$, there exists a sequence of Kobayashi chains joining x to y such that the length of the chains tends to zero. Because π is a *finite* covering, and therefore there are only finitely many points in $\widehat{X}$ lying above y, infinitely many of these chains must lift to Kobayashi chains joining $\hat{x}$ to a fixed lift $\hat{y}$ of y. Therefore, $d(\hat{x}, \hat{y}) = 0$. Because $\widehat{X}$ is assumed not to be Kobayashi anti-hyperbolic, this implies $\hat{x} = \hat{y}$. Therefore, $x = y$, and X cannot be Kobayashi anti-hyperbolic either.

Remark. In the next section when we study Abelian varieties, we will have to consider topological covering maps which are infinite covers. In that case, the above argument will not work, but because we have very specific knowledge about the structure of the spaces involved, we will be able to show what we want. It would be nice to answer the following question in general: *Is it true that an analytic space is Kobayashi anti-hyperbolic if and only if its universal cover is Kobayashi anti-hyperbolic?*

Caution! I should remark at this point that an étale morphism over the complex numbers is also a topological covering map. This is definitely not the case p-adically. For example, let A and A' be two Abelian varieties with Abelian reduction. Any non-trivial isogeny from A to A' will be an étale morphism in the sense of algebraic geometry. However, it will not be a covering map of the underlying Berkovich topological spaces. Indeed, if a' is the unique point in A' lying above the generic point of the reduction $\widetilde{A}'$, then the only point in the inverse image of a' is the unique point of A lying above the generic point of $\widetilde{A}$.

Lemma 2.12. *Let X be a formal analytic space with reduction $\widetilde{X}$. Assume that $\widetilde{X}$ does not contain any rational curves. Let Y be an arc-connected analytic subspace of $\mathbf{P}^1$, and let $f: Y \to X$ be an analytic map. Then, the image of f lies above a single closed point of $\widetilde{X}$.*

Proof: First we show that the image of f lies entirely above the closed points in $\widetilde{X}$. Indeed, suppose there exists a point $y \in Y$ such that the point $x = f(y) \in X$ is such that $\tilde{x}$ is not a closed point of $\widetilde{X}$. Then, there would be a non-zero homomorphism from $\mathcal{K}(x)$ into $\mathcal{K}(y)$. Proposition IV.4.5 would then imply

$$\mathcal{K}(\tilde{x}) \hookrightarrow \widetilde{\mathcal{K}(x)} \hookrightarrow \widetilde{\mathcal{K}(y)}.$$

However by Proposition IV.1.3, $\widetilde{\mathcal{K}(y)}$ is either $\mathbf{F}_p^a$ or the field of rational functions in one variable over $\mathbf{F}_p^a$, but since $\tilde{x}$ is assumed not to be closed, $\widetilde{\mathcal{K}(y)}$ must be the rational function field. This then gives a non-constant map from $\mathbf{P}^1_{\mathbf{F}_p^a} \to \widetilde{X}$, contradicting the assumption that $\widetilde{X}$ contains no rational curves. Now since Y is arc-connected, $f(Y)$ is also arc-connected. Proposition IV.4.6 then implies that the image of f cannot lie above more than one closed point in $\widetilde{X}$.

Lemma 2.13. *Let X be a formal analytic space with reduction $\widetilde{X}$. Assume that $\widetilde{X}$ is smooth and does not contain any rational curves. Then, X is not Kobayashi anti-hyperbolic.*

Proof: By Lemma 2.12, $f: \mathbf{B} \to X$ must lie entirely above a single closed point $\tilde{x} \in \widetilde{X}$ because $\widetilde{X}$ is assumed not to contain any rational curves. Therefore, any Kobayashi chain in X lies above a single closed point $\tilde{x} \in \widetilde{X}$. Since $\widetilde{X}$ is smooth, the inverse image of $\tilde{x}$ is isomorphic to $\mathring{\mathbf{B}}^n$ by Proposition IV.4.6, so X is not Kobayashi anti-hyperbolic by Example 2.7.

3. Abelian Varieties and Algebraic Surfaces. This section investigates the anti-hyperbolicity of some special cases.

Abelian Varieties. Theorem VI.4.2 said that any analytic map from $\mathbf{A}^1$ into an Abelian variety must be a constant. In other words, Abelian varieties are not Brody anti-hyperbolic. My first goal in this section is to show that Abelian varieties are also not Kobayashi anti-hyperbolic – that is the Kobayashi semi-distance on an Abelian variety is a genuine distance. The proof of this fact will essentially involve showing that neither Abelian varieties with good reduction nor compact tori (the totally degenerate reduction case) are Kobayashi anti-hyperbolic, and then applying the uniformization theorem for Abelian varieties to conclude the general case.

Example 3.1. *The analytic space $\mathbf{A}^{1\times}$ is not Kobayashi anti-hyperbolic. Furthermore, if $x, y \in \mathbf{A}^{1\times}(\mathbf{C}_p)$ are such that $|x|_p \neq |y|_p$, then $d(x,y) = \infty$.*

Proof: Let $x \in \mathbf{A}^{1\times}(\mathbf{C}_p)$, and let $f\colon \mathbf{B} \to \mathbf{A}^{1\times}$ be an analytic map such that x is in the image of f. Write $f = \sum_{k=0}^{\infty} c_k z^k$. Now since x is in the image of f, we have $\sup_k |c_k|_p \geq |x|_p$. On the other hand, since f does not have a zero, its valuation polygon cannot have any critical points, and therefore $|c_k|_p < |c_0|_p$ for all $k > 0$. Therefore, $|f(z)|_p = |c_0|_p = |x|_p$ for all $z \in \mathbf{B}(\mathbf{C}_p)$. This implies that if $x, y \in \mathbf{A}^{1\times}(\mathbf{C}_p)$ are such that $|x|_p \neq |y|_p$, then there cannot be a Kobayashi chain joining x to y, and hence $d(x,y) = \infty$. Furthermore, even if $|x|_p = |y|_p = r$, any Kobayashi chain joining x to y must be contained in $\mathbf{B}(r)$, the ball of radius r. Therefore, if we could find Kobayashi chains in $\mathbf{A}^{1\times}$ of arbitrarily small length joining x to y, then we could find Kobayashi chains in $\mathbf{B}(r)$ of arbitrarily small length joining x to y. But, by Example 2.2, $\mathbf{B}(r) \cong \mathbf{B}$ is not Kobayashi anti-hyperbolic, so x must in fact equal y, and hence $\mathbf{A}^{1\times}$ is also not Kobayashi anti-hyperbolic.

Applying Proposition 2.6 to the above, we get

Example 3.2. *If T is an affine analytic torus, then T is not Kobayashi anti-hyperbolic.*

Later we will also need the following fact about the Kobayashi semi-distance on affine tori.

Lemma 3.3. *Let T be an affine analytic torus and Γ a discrete, torsion free, subgroup of $T(\mathbf{C}_p)$. Let $t \in T(\mathbf{C}_p)$, and let γ be a non-trivial element in Γ. Then, $d(t, \gamma t) = \infty$.*

Proof: Write $T = \mathbf{G}_m \times \cdots \times \mathbf{G}_m$, and write $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $t = (t_1, \ldots, t_n)$. Because T is a product space, Proposition 2.6 says that

$$d(t, \gamma t) \geq \max_{1 \leq j \leq n} d(t_j, \gamma_j t_j),$$

where the Kobayashi semi-distance on the right is the Kobayashi semi-distance on $\mathbf{G}_m \cong \mathbf{A}^{1\times}$. Now since γ is not the identity, there is at least one j such that $|\gamma_j|_p \neq 1$. Indeed, if Γ had an element $\gamma = (\gamma_1, \ldots, \gamma_n)$ other than the identity such that $|\gamma_i|_p = 1$ for all i, then either γ would be a torsion element, or the subgroup generated by γ would have an accumulation point in the Berkovich analytic space

T because it would lie in a compact subset of T. This would then contradict the assumption that Γ is discrete and torsion free. Therefore, there is at least one j such that $|t_j|_p \neq |\gamma_j t_j|_p$. Then by Example 3.1, $d(t_j, \gamma_j t_j) = \infty$, and we are done.

Example 3.4. *If X is a complete analytic torus (not necessarily algebraic), then X is not Kobayashi anti-hyperbolic.*

Proof: By assumption, $X = T/\Gamma$, where T is an affine analytic torus and Γ is a discrete, torsion free, subgroup of $T(\mathbf{C}_p)$ with rank equal to the dimension of T. Furthermore, the natural map $T \to X$ is a topological covering map. Let d denote the Kobayashi semi-distance on X, and let $\hat{d}$ denote the Kobayashi semi-distance on T. Let x, y be two points in $X(\mathbf{C}_p)$ such that $d(x, y) = 0$. Fix a lift $\hat{x}$ of x in T. Any Kobayashi chain joining x to y can be lifted to a Kobayashi chain of the same length joining $\hat{x}$ to some lift $\hat{y}$ of y because $T \to X$ is a covering map. *A priori,* the lifts of two different Kobayashi chains joining x to y may lift to chains joining $\hat{x}$ to two different lifts $\hat{y}$. However, Lemma 3.3 tells us that $\hat{d}(\hat{y}_1, \hat{y}_2) = \infty$ for two different lifts of y, so the triangle inequality then implies that all lifts of Kobayashi chains joining x to y lift to chains joining $\hat{x}$ to the same $\hat{y}$. Therefore, $\hat{d}(\hat{x}, \hat{y}) = 0$, so by Example 3.2, $\hat{x} = \hat{y}$. Therefore, $x = y$, and we are done.

Lemma 3.5. *Let*

$$1 \longrightarrow T \longrightarrow G \xrightarrow{\ \phi\ } B \longrightarrow 1$$

be an algebraic extension of an Abelian variety B with good reduction by an affine analytic torus T. Let $\pi: B \to \widetilde{B}$ be the reduction map for B. Let $\tilde{b}$ be a closed point in $\widetilde{B}$. Let $U = \pi^{-1}(\tilde{b}) \cong \mathring{\mathbf{B}}^n$ be the open set in B lying above $\tilde{b}$. Then, there is an analytic isomorphism

$$\mathring{\mathbf{B}}^n \times T \cong U \times T \cong \phi^{-1}(U).$$

Proof: Because all of the arrows in the above short exact sequence are algebraic morphisms and because T is a connected, solvable, algebraic group, we can apply Theorem 10 of [**Ro**] to conclude that there exists a non-empty Zariski open subset V of B and an algebraic morphism $\sigma: V \to G$ such that $\phi \circ \sigma = \mathrm{id}$.

Now, I claim that there exists a closed point $\tilde{b}_0$ in $\widetilde{B}$ such that $\pi^{-1}(\tilde{b}_0) \subset V$. To see this, let Z be the complement of V in B, and let $W = \mathcal{M}(A)$ be a small affinoid neighborhood in B compatible with the reduction $\pi: B \to \widetilde{B}$ such that there exist $f_1, \ldots, f_r \in A$ so that

$$W \cap Z = \{w \in W : f_1(w) = \cdots = f_r(w) = 0\}.$$

Since V is not empty, we may assume that none of the f_i are identically zero. By multiplying each f_i by a non-zero constant, we may also assume that $|f_i|_{\sup} = 1$

for all i. Let

$$D(\tilde{f}_i) = \{\tilde{b} \in \widetilde{W} \subset \widetilde{B} : \tilde{f}_i(\tilde{b}) \neq 0\}.$$

Then, by Lemma IV.4.3,

$$\pi^{-1}(D(\tilde{f}_i)) = \{b \in W : |f_i(b)|_{\sup} = 1\}.$$

Now if for every $\tilde{b}$, there is at least one b above $\tilde{b}$ such that $f(b) = 0$, then this implies that $\pi^{-1}(D(\tilde{f}_i)) = \emptyset$, and hence $D(\tilde{f}_i) = \emptyset$. Therefore, $\tilde{f}_i = 0$ for all i, and hence $f_i \equiv 0$ for all i, contradicting the assumption that V is not empty. Thus, there is indeed a closed point $\tilde{b}_0$ in $\widetilde{B}$ with $\pi^{-1}(\tilde{b}_0)$ contained in V.

Next, let $\tilde{b}$ be any closed point of $\widetilde{B}$. Because $\widetilde{B}$ is a group, translation by the appropriate group element gives us an automorphism $\tilde{\tau}$ such that $\tilde{\tau}(\tilde{b}) = \tilde{b}_0$. Now, $\tilde{\tau}$ comes from an automorphism τ given by a translation on B. Therefore, $\sigma \circ \tau|_{\pi^{-1}(\tilde{b})}$ gives the isomorphism

$$\phi^{-1}(\pi^{-1}(\tilde{b})) \cong \pi^{-1}(\tilde{b}) \times T.$$

The proof of the lemma is therefore completed by recalling that $\pi^{-1}(\tilde{b}) \cong \mathring{\mathbf{B}}^n$ by Proposition IV.4.6 since all closed points of $\widetilde{B}$ are smooth.

Theorem 3.6. *If A is an Abelian variety, then A is not Kobayashi anti-hyperbolic.*

Proof: Let G be the universal cover of A, and let

$$1 \longrightarrow T \longrightarrow G \xrightarrow{\ \phi\ } B \longrightarrow 1$$

be the exact sequence from Theorem VI.3.4. Also, let Γ be the discrete group from Theorem VI.3.4. Recall that B has good reduction, and let $\pi: B \to \widetilde{B}$ denote the reduction map.

First we will see that G is not Kobayashi anti-hyperbolic. Let $x, y \in G(\mathbf{C}_p)$ be two points such that $d(x,y) = 0$. If f is any analytic map from $\mathbf{B}$ into G, then by Lemma 2.12, the image of $\phi \circ f$ in B must lie entirely above a single closed point $\tilde{b}$ of $\widetilde{B}$ because $\widetilde{B}$ does not contain any rational curves. Let $\tilde{b}$ denote this closed point, and let $U = \pi^{-1}(\tilde{b})$. Therefore, any Kobayashi chain connecting x to y in G must be entirely contained in $\phi^{-1}(U)$. Hence, it suffices to verify that $\phi^{-1}(U)$ is not Kobayashi anti-hyperbolic. By Lemma 3.5,

$$\phi^{-1}(U) \cong \mathring{\mathbf{B}}^n \times T.$$

Therefore, since neither $\mathring{\mathbf{B}}^n$ nor T is Kobayashi anti-hyperbolic, Proposition 2.6 implies that $\phi^{-1}(U)$ is also not Kobayashi anti-hyperbolic, and we have therefore shown that G is not Kobayashi anti-hyperbolic.

Now, let x be in $A(\mathbf{C}_p)$, and let $\hat{x}$ and $\hat{x}'$ be two different lifts of x in G. We will show that $\hat{d}(\hat{x},\hat{x}')=\infty$. If $\hat{d}(\hat{x},\hat{x}')\neq\infty$, then they can be joined by a Kobayashi chain, so must lie above a single closed point $\tilde{b}$ in $\widetilde{B}$ as above. By Lemma 3.5, we can then consider $\hat{x}$ and $\hat{x}'$ to be points in $\mathring{\mathbf{B}}^n\times T$. Furthermore, since we have assumed that $\hat{x}$ and $\hat{x}'$ are lifts of the same point in A, their images under the projection $\mathring{\mathbf{B}}^n\times T\to T$ differ by translation by an element of Γ. Proposition 2.6 then says that $\hat{d}(\hat{x},\hat{x}')$ is greater than the Kobayashi distance between the images of $\hat{x}$ and $\hat{x}'$ in T, which is equal to infinity by Lemma 3.3.

Finally, let x and y be two points in $A(\mathbf{C}_p)$ such that $d(x,y)=0$. Fix a lift $\hat{x}$ of x in G. As in Example 3.4, any Kobayashi chain joining x to y in A will lift to a Kobayashi chain of the same length joining $\hat{x}$ to some lift $\hat{y}$ of y in G. From above, the Kobayashi distance between two different lifts in G of y is infinite, so by the triangle inequality, as in Example 3.4, every lift of a Kobayashi chain joining x to y lifts to a Kobayashi chain joining $\hat{x}$ to the same lift $\hat{y}$. Therefore, $\hat{d}(\hat{x},\hat{y})=0$, and because G is not Kobayashi anti-hyperbolic, $\hat{x}=\hat{y}$. Therefore, $x=y$ and the proof of the theorem is complete.

Algebraic Curves. If X is a smooth projective curve of positive genus, then X can be embedded in its Jacobian, so we get

Theorem 3.7. *Let X be an irreducible algebraic curve, and let $\widehat{X}$ be its normalization. Then, X is Kobayashi anti-hyperbolic if and only if $\widehat{X}\cong\mathbf{P}^1$ or $\widehat{X}\cong\mathbf{A}^1$.*

Proof: If $\widehat{X}\cong\mathbf{P}^1$ or $\widehat{X}\cong\mathbf{A}^1$, then X is clearly Kobayashi anti-hyperbolic. To show the converse, assume that $\widehat{X}\ncong\mathbf{P}^1$ and that $\widehat{X}\ncong\mathbf{A}^1$. Because $\widehat{X}$ is a normal curve, it is smooth by Fact IV.5.1. If $\widehat{X}$ has positive genus, then it is not Kobayashi anti-hyperbolic because it is an analytic subspace of its projective completion, which is not Kobayashi anti-hyperbolic because it is a subvariety of its Jacobian, which is not Kobayashi anti-hyperbolic by Theorem 3.6. If $\widehat{X}$ has genus 0, then $\widehat{X}\subseteq\mathbf{A}^{1\times}$ because of the assumption that $\widehat{X}\ncong\mathbf{P}^1$ or $\mathbf{A}^1$. Therefore, $\widehat{X}$ is not Kobayashi anti-hyperbolic by Example 3.1. Any non-constant analytic map $\mathbf{B}\to X$ satisfies the hypotheses of Fact IV.5.3, so any Kobayashi chain in X will lift to a Kobayashi chain in $\widehat{X}$ of the same length. Since the normalization map $\widehat{X}\to X$ is a finite morphism, we see, as in Lemma 2.11, that the fact that $\widehat{X}$ is not Kobayashi anti-hyperbolic implies that X is not Kobayashi anti-hyperbolic either, and we are done.

Algebraic Surfaces. Next I will use the classification of algebraic surfaces to give some support to the idea that projective algebraic varieties should be Brody anti-hyperbolic if and only if they are Kobayashi anti-hyperbolic, and that both of these conditions are equivalent to the existence of a non-constant algebraic morphism from $\mathbf{P}^1$ into the variety (the image of such a morphism is called a **rational curve**). Of course, any variety that contains a rational curve will be both Brody and Kobayashi anti-hyperbolic. I also give some support for Conjecture VI.4.7 which said that smooth projective varieties should be (p-adic) Brody hyperbolic if and only if every subvariety is either of general type or the image under an algebraic morphism of an Abelian variety with Abelian reduction. Of course, anything which is Brody anti-hyperbolic is not Brody hyperbolic. A. Beauville's [**Bea**] book is an excellent reference for the classification of algebraic surfaces. For the rest of this section, surface will mean a smooth projective algebraic surface. I proceed by Kodaira dimension.

Kodiara dimension $-\infty$. By the classification theorems, all surfaces of Kodaira dimension $-\infty$ contain lots of rational curves. Therefore, all surfaces of Kodaira dimension $-\infty$ are both Brody and Kobayashi anti-hyperbolic.

Kodaira dimension 0. The classification theorem puts minimal models of surfaces of Kodaira number 0 into four categories. (If a surface is not minimal, then it contains a rational curve and is therefore anti-hyperbolic.)

K3-Surfaces. By the appendix to Mori-Mukai [**M-M**], completing the work of Green-Griffiths [**G-G**], all K3-surfaces contain at least one rational curve, and hence are both Brody and Kobayashi anti-hyperbolic.

Enriques surfaces. An Enriques surface can always be realized as a finite quotient of a K3-surface, so Enriques surfaces must also always contain at least one rational curve. Therefore, these surfaces are also both Brody and Kobayashi anti-hyperbolic.

Abelian surfaces. Abelian surfaces contain no rational curves, Theorem VI.4.2 shows that Abelian surfaces are not Brody anti-hyperbolic, and Theorem 3.6 above shows that Abelian surfaces are also not Kobayashi anti-hyperbolic.

Bi-elliptic (or hyperelliptic) surfaces. If S is a bi-elliptic surface, then there exist two elliptic curves, E and F and a finite group G which acts on E by translation and on F by group automorphisms and possibly translations. Thus, there is a finite étale map

$$\Phi: E \times F \to S.$$

Note that this implies that S does not contain any rational curves, because rational curves lift to étale covers, and $E \times F$, being an Abelian variety, does not contain any rational curves.

Theorem 3.8. *Let S be a bi-elliptic surface. Let E and F be the elliptic curves and G be the finite group which give S as $(E \times F)/G$. Then, S is not Brody anti-hyperbolic, and S is Brody hyperbolic if and only if both E and F have good reduction.*

Proof: There is an elliptic fibration $S \to E'$ over $E' = E/G$ with fiber F.

In the case that either E or F has bad reduction, then S contains the rational image of an elliptic curve with bad reduction and cannot be Brody hyperbolic. If both E and F have good reduction, then S is Brody hyperbolic as follows. Let $f\colon \mathbf{A}^{1\times} \to S$ be an analytic map. Then, composing f with the fibration of S over E' above, gives an analytic map of $\mathbf{A}^{1\times}$ into E, which must be constant since E' has good reduction. Therefore, the image of f in S lies in a single fiber over E'. However, all of these fibers are isomorphic to F, which also has good reduction, so f must in fact be constant; therefore, S is Brody Hyperbolic.

Similarly, we see that S is not Brody anti-hyperbolic (regardless of whether E and F have good reduction) because any analytic map from $\mathbf{A}^1$ into S must lie above a single point in E', and since it is therefore contained in a fiber isomorphic to the elliptic curve F, it must then be constant. Hence the theorem.

Conjecturally, bi-elliptic surfaces S should also not be Kobayashi anti-hyperbolic. However, here the situation is more subtle. I split the discussion into two cases, depending on whether the reduction of E and F is good or bad.

Case 1: It sometimes happens that the étale map $\Phi\colon E \times F \to S$ above is in fact a topological covering map for the underlying Berkovich topological spaces. Because the map Φ is finite, in this case Lemma 2.11 can be applied to conclude that S is not Kobayashi anti-hyperbolic.

In order for Φ to be a topological covering map, it is necessary that E have bad reduction. Indeed, if E has good reduction, then the action of G on E always fixes the unique point in E lying above the generic point in $\widetilde{E}$. However, that E have bad reduction is not a sufficient condition for Φ to be a topological covering map. Let's examine the special case when $G \cong \mathbf{Z}/2\mathbf{Z}$ acts on E by translation by a 2-torsion point. The action of G on $E \times F$ lifts to an action on $\mathbf{G}_m \times F$ because the assumption that E have bad reduction implies $E \cong \mathbf{G}_m/ < q >$, with $|q|_p \neq 1$. This lifted operation must still act by translation, so one of two things can happen. Either the non-trivial element in G acts by $z \mapsto -z$, or by $z \mapsto \pm\sqrt{q}z$. In the first case, G has fixed points – for instance, the point corresponding to the multiplicative semi-norm

$$f \mapsto \sup\{|f(z)|_p : z \in \mathbf{C}_p, |z|_p = 1\}.$$

In this case, Φ is not a topological covering map. Whereas in the other case, it is easy to see that G acts fixed point free, so Φ is a topological covering map.

Case 2: If E and F both have good reduction, then they can be regarded as formal analytic groups. In this case, G acts on $E \times F$ in such a way as to preserve the formal structure. To see this, note that G acts on E by translation, so must preserve the formal structure by definition. There are essentially only three ways an element γ of G can act on F, other than by translation. One possibility is that γ

act by involution, and this must preserve the formal structure on G by definition. Another possibility is that F is given by the equation $y^2 = x^3 - 1$, and γ acts by $(x,y) \mapsto (\rho x, y)$, where ρ is a third root of unity. The formal structure here comes from the standard affinoid cover of $\mathbf{P}^2$, and since $|\rho|_p = 1$, γ preserves this structure. The final possibility is when F is given by the equation $y^2 = x^3 - x$, and γ acts by $(x,y) \mapsto (-x, \pm\sqrt{-1}y)$. Again, one sees explicitly that γ preserves the formal structure on F, which again comes from the standard affinoid cover of $\mathbf{P}^2$.

Therefore, Φ induces a formal analytic structure on S, and one gets a map

$$\widetilde{\Phi}: \widetilde{E} \times \widetilde{F} \to \widetilde{S}.$$

Now, it might happen that G acts on $\widetilde{E} \times \widetilde{F}$ without fixed points, so that $\widetilde{\Phi}$ is étale. In this case, $\widetilde{S}$ does not contain any rational curves, so Lemma 2.13 implies S is not Kobayashi anti-hyperbolic. However, if $p = 2$ or 3, then it is possible that G might not act without fixed points on $\widetilde{E} \times \widetilde{F}$, and hence $\widetilde{S}$ might contain a rational curve. For instance, suppose that G acts on E by translation by a 2-torsion point. Assume also that $p = 2$ and that this 2-torsion point reduces to the origin (this will always happen if $\widetilde{E}$ is supersingular). In this case, G will act on $\widetilde{E}$ only by the identity, so that

$$\widetilde{S} \cong \widetilde{E} \times \mathbf{P}^1_{\mathbf{F}_2^a}.$$

Clearly, this treatment of the Kobayashi semi-distance on bi-elliptic surfaces is less than satisfactory. I have included it here because I wanted to give the reader an idea of the different kinds of things that can happen. To give a more satisfactory treatment of bi-elliptic surfaces, one needs to investigate how étale covers, which need not be topological covers, affect the Kobayashi semi-distance. Note that there are lots of non-trivial étale covers of the unit ball $\mathbf{B}$. For instance, Robert Coleman pointed out to me that if d is prime to p, then the space

$$Y = \mathcal{M}(\mathbf{C}_p < y, z > /(y^p + y - z^d))$$

is an affinoid open set in a curve of genus $(p-1)(d-1)/2$ and gives rise to a degree p étale cover of $\mathbf{B}$ by projection to the z coordinate. These examples make it difficult to see how étale covers might affect the Kobayashi semi-distance because Kobayashi chains might not lift to étale covers of analytic spaces.

Positive Kodaira Dimension. Here we proceed according to the irregularity – recall that the irregularity of a smooth projective algebraic variety is by definition the dimension of $H^1(X, \mathcal{O}_X)$. Unfortunately, I cannot yet say anything about whether surfaces with positive Kodaira dimension are or are not Kobayashi anti-hyperbolic if they do not contain rational curves, so I restrict myself to Brody anti-hyperbolicity for the rest of this section.

Irregularity 0. This is the most interesting case because hypersurfaces in $\mathbf{P}^3$ have irregularity zero. However, this is also the case that I cannot say anything about. All the results here are somehow tied to Abelian varieties, and varieties with irregularity zero do not map in any non-trivial way to an Abelian variety.

Positive Irregularity. If S has positive irregularity, then the Albanese morphism $\alpha: S \to A$ maps S in a non-constant way into an Abelian variety A of positive dimension. If $f: \mathbf{A}^1 \to S$ is an analytic map, then Theorem VI.4.2 says that $\alpha \circ f$ must be constant. Hence, the image of f lies in an irreducible component of a fiber of α, which is a (possibly singular) projective algebraic curve. Theorem V.4.1 tells us that either that f is constant or that this fiber contains a rational curve. Therefore, surfaces with positive irregularity are Brody anti-hyperbolic if and only if they contain rational curves.

Irregularity > 2. If, in addition, S has irregularity greater than two, Theorem VI.4.5 tells us that the image of any analytic map from $\mathbf{A}^{1\times}$ into S must be contained in a subvariety (i.e. a possibly singular irreducible algebraic curve). Again, Theorem V.4.1 tells us either that f must be constant or that the normalization of the curve is either $\mathbf{P}^1$ or an elliptic curve with bad reduction.

Kodaira Dimension 1. By Kodaira's classification, all surfaces of Kodaira dimension 1 are elliptic fibrations over a curve. That means there exists a surjective morphism $f: S \to C$ from S to a curve C such that the generic fiber is a smooth irreducible curve of genus 1. The following proposition, whose proof was given to me by Geoffrey Mess and Najmuddin Fakhruddin, tells us that any such surface without rational curves has positive irregularity and is hence Brody anti-hyperbolic by the above.

Proposition 3.9. *Let $f: S \to C$ be a surjective morphism from an algebraic surface to an algebraic curve such that the generic fiber is a smooth irreducible curve of genus 1. Then, at least one of the following two conditions is satisfied.*

(*a*) *S contains a rational curve.*

(*b*) *S has positive irregularity.*

Proof: Let χ_{top} denote the topological Euler characteristic of a complex space. Since S is compact when considered as a complex space, we have from Proposition 11.4 on page 97 of [**BPV**] that

$$\chi_{\text{top}}(S) = \chi_{\text{top}}(F_{\text{gen}})\chi_{\text{top}}(C) + \sum_j [\chi_{\text{top}}(F_j) - \chi_{\text{top}}(F_{\text{gen}})],$$

where F_{gen} denotes the generic fiber and the sum on the right is the sum over all the singular and multiple fibers. Now since the fibration is by curves of genus 1, $\chi_{\text{top}}(F_{\text{gen}})$ is zero. If S is known not to contain any rational curves, then smooth multiple fibers are the only possible degenerate fibers. In this case, $\chi_{\text{top}}(F_j)$ is also equal to zero for all j. Therefore, such a surface has topological Euler characteristic equal to zero, and hence a positive first Betti number. Since S is algebraic, this implies that the Albanese variety of S is positive dimensional, so if S does not contain any rational curves, then property (b) in the statement of the proposition will be satisfied.

In summary, we have

Theorem 3.10. *Let S be a smooth algebraic surface over $\mathbf{C}_p$. (1) If either the Kodaira dimension of S is ≤ 1 or the irregularity of S is ≥ 1, then S is Brody anti-hyperbolic if and only if S contains a rational curve. (2) If either the Kodaira dimension of S is ≤ 1 or the irregularity of S is > 2, then S is Brody hyperbolic if and only if S contains neither rational curves nor Abelian varieties with non-Abelian reduction. (3) If S has Kodaira dimension equal to $-\infty$, or if S has Kodaira dimension 0 but is not a bi-elliptic surface, then S is Kobayashi anti-hyperbolic if and only if S contains a rational curve, which is always the case unless S is an Abelian surface.*

4. Open Questions. Where does that leave us? The motivation behind Lang's conjecture stated in the introduction is that holomorphic curves in projective varieties come from rational images of group varieties. Rational images of group varieties in projective varieties come either from rational images of Abelian varieties or from rational images of $\mathbf{P}^1$. On the p-adic side, we have shown that p-adic analytic maps from $\mathbf{A}^1$ cannot come from Abelian varieties, so that leaves only rational curves. This together with the examples above leads me to the following conjecture:

Conjecture 4.1. *Let X be a non-singular projective algebraic variety defined over $\mathbf{C}_p$. Then, X is Brody anti-hyperbolic if and only if X contains a rational curve.*

A weaker conjecture that may help us understand the above conjecture is

Conjecture 4.2. *Let X be a non-singular projective algebraic variety defined over a finite extension L of $\mathbf{Q}_p$. Then, X is Brody anti-hyperbolic if and only if there exists a non-constant analytic map $f\colon \mathbf{A}^1 \to X$ defined over some finite extension L' of L. The latter condition means that we can embed X into projective space so that f can be represented by projective coordinate functions which are given by power-series, all of whose coefficients lie in L'.*

Of course, Conjecture 4.1 implies Conjecture 4.2 since any algebraic map satisfies the second condition in 4.2, but approximation techniques might make Conjecture 4.2 a more tractable problem given our current knowledge of algebraic geometry.

That leaves us with Kobayashi anti-hyperbolicity. The proof of Brody's Theorem over the complex numbers makes use of compactness in an essential way. In fact, there is an example (see Example 3 on page 79 of [**L1**].) to show that the theorem can be false without the compactness assumption. Therefore, I am hesitant to make any conjectures about the truth of the p-adic analogue over non-locally compact fields. I am, however, hopeful that projectivity can substitute for the compactness condition, and yield a positive answer to the following question. Of course, another approach is to use something like an affirmative answer to Conjecture 4.2 to prove an analogue of Brody's theorem over a finite extension of $\mathbf{Q}_p$.

Question 4.3. *If X is a non-singular projective algebraic variety defined over $\mathbf{C}_p$, is it true that X is Brody anti-hyperbolic if and only if X is Kobayashi anti-hyperbolic?*

Finally, let me recall Conjectures VI.4.6 and VI.4.7:

Conjecture VI.4.6. *Let X be a projective variety defined over $\mathbf{C}_p$. If X is of general type, then the image of any analytic map $f: \mathbf{C}_p^\times \to X$ is contained in a proper subvariety of X.*

Conjecture VI.4.7. *Let X be a projective variety defined over $\mathbf{C}_p$. If every subvariety of X (including X itself) is of general type, then X is p-adic Brody hyperbolic. Furthermore, if X is p-adic Brody hyperbolic, then either every subvariety of X (including X itself) is of general type, or there exists an Abelian variety A with Abelian reduction and a non-constant algebraic morphism $\phi: A \to X$.*

REFERENCES

[Am] Y. AMICE, *Les nombres p-adiques*, Presses Universitaires de France, 1975.

[Bea] A. BEAUVILLE, *Complex Algebraic Surfaces*, London Mathematical Society Lecture Notes Series **68**, Cambridge University Press, 1983.

[Ber] V. BERKOVICH, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, AMS Surveys and Monographs **33**, 1990.

[Ber2] V. BERKOVICH, *Étale Cohomology for Non-Archimedean Analytic Spaces*, draft 1993.

[BGR] S. BOSCH, U. GÜNTZER and R. REMMERT, *Non-Archimedean Analysis*, Springer-Verlag, 1984.

[BL1] S. BOSCH and W. LÜTKEBOHMERT, *Stable Reduction and Uniformization of Abelian Varieties* I, Math. Ann. **270** (1985), 349-379.

[BL2] S. BOSCH and W. LÜTKEBOHMERT, *Stable Reduction and Uniformization of Abelian Varieties* II, Invent. Math. **78** (1984), 257-297.

[Bos] S. BOSCH, *Rigid Analytische Gruppen mit Guter Reduction,* Math. Ann. **223** (1976), 193-205.

[Bou] A. BOUTABAA, *Theorie de Nevanlinna p-Adique*, Manuscripta Math. **67** (1990), 251-269.

[BPV] W. BARTH, C. PETERS, and A. VAN DE VEN, *Compact Complex Surfaces*, Springer-Verlag, 1984.

[Br] R. BRODY, *Compact Manifolds and Hyperbolicity*, Trans. Amer. Math. Soc. **235** (1978), 213-219.

[F-P] J. FRESNEL and M. VAN DER PUT, *Géométrie Analytique Rigide et Applications*, Birkhäuser, 1981.

[G-G] M. GREEN and P. GRIFFITHS, *Two Applications of Algebraic Geometry to Entire Holomorphic Mappings*, The Chern Symposium 1979, Proc. International Sympos., Berkeley, CA, 1979, Springer-Verlag, 1980, pp. 41-74.

[G-P] L. GERRITZEN and M. VAN DER PUT, *Schottky Groups and Mumford Curves,* Lecture Notes in Mathematics **817**, Springer-Verlag, 1980.

[G-R] H. GRAUERT and R. REMMERT, *Coherent Analytic Sheaves,* Springer-Verlag, 1984.

[Har] R. HARTSHORNE, *Algebraic Geometry,* Springer-Verlag, 1977.

[Hay] W. K. HAYMAN, *Meromorphic Functions*, Oxford University Press, 1964.

[KQ] HÀ HUY KHOÁI and MY VINH QUANG, *On p-adic Nevanlinna Theory*, in Lecture Notes in Mathematics **1351**, Springer-Verlag 1988.

[K1] HÀ HUY KHOÁI, *On p-Adic Meromorphic Functions*, Duke Mathematical Journal. **50** (1983), 695-711.

[K2] HÀ HUY KHOÁI, *Heights for p-Adic Meromorphic Functions and Value Distribution Theory*, Max-Planck-Institut Für Mathematik. **89-67** (1989).

[K3] HÀ HUY KHOÁI, *Heights for p-Adic Holomorphic Functions of Several Variables*, Max-Planck-Institut Für Mathematik. **89-83** (1989).

[L1] S. LANG, *Introduction to Complex Hyperbolic Spaces,* Springer-Verlag, 1987.

[L2] S. LANG, *Number Theory* III, Encyclopedia of Mathematical Sciences **60**, Springer-Verlag, 1991.

[L3] S. LANG, *Fundamentals of Diophantine Geometry,* Springer-Verlag, 1983.

[Lü] W. LÜTKEBOHMERT, *Der Satz von Remmert-Stein in der nichtarchimedischen Funktionentheorie,* Math. Z. **139** (1974), pp. 69-84.

[M-M] S. MORI and S. MUKAI, *The Uniruledness of the Moduli Space of Curves of Genus 11*, Algebraic Geometry Conference, Tokyo-Kyoto, 1982, in Lecture Notes in Mathematics **1016**, Springer-Verlag, 1983, pp. 334-353.

[Ro] M. ROSENLICHT, *Some Basic Theorems on Algebraic Groups,* American Journal of Mathematics **78** (1956), pp. 401-443.

[Ru] M. RU, *The Geometric and Arithmetic Aspects of* $\mathbf{P}^n$ *Minus Hyperplanes,* preprint, 1992.

[Sch] P. SCHNEIDER, *Points of Rigid Analytic Varieties,* J. reine angew. Math., **434** (1993), pp. 127-157

[Si] J. H. SILVERMAN, *Arithmetic Distance Functions and Height Functions in Diophantine Geometry,* Math. Ann., **279** (1987), pp. 193-216.

[Ta] J. TATE, *Rigid Analytic Spaces,* Invent. Math. **12** (1971), 257-289.

[Vo] P. VOJTA, *A Refinement of Schmidt's Subspace Theorem,* American Journal of Mathematics **111** (1989), pp. 489-518.

[We] L. WENG, *From Subvarieties of Abelian Varieties to Kobayashi Hyperbolic Manifolds: after P. Vojta and G. Faltings*, preprint 1992.

INDEX