Good & Bad Gluing

Good Gluing. Let $\mathbf{R}$ denote the real numbers with its usual topology. Let $X = Y = \mathbf{R}$. Let $A = X \setminus \{0\} \subset X$. Let $f : A \to Y$ be the continuos map $f(x) = 1/x$. Let $Z = Y \cup_f X$, as in Definition I.13.13. Verify that Z is a manifold, and is in fact homeomorphic to $\mathbf{S}^1$. Explain how you can view this construction as gluing two real lines together to create a circle.

Bad Gluing. Let X, Y, and A be as in the example above, but this time consider $g : A \to Y$ to be the continous map $g(x) = x$. Let $W = Y \cup_g X$. Check that W is a second countable topological space such that every point has an open neighborhood homeomorphic to an open subset of $\mathbf{R}$. However, verify that W is *not* Hausdorff, and therefore not a manifold. Moreover, let 0_X denote the point in W that corresponds to $0 \in X$, let 0_Y denote the point in W that corresponds to 0 in Y, let U be an open set in W the contains both 0_X and 0_Y, and let h be a continuous function from U to $\mathbf{R}$. Then, show it must be that $h(0_X) = h(0_Y)$. This is annoying. In particular, explain why this means there can be no injective continous map from W to $\mathbf{R}^N$, no matter how large N is. Compare with Theorem II.10.7. In fact, Lemma II.10.3 fails for W. The Hausdorff condition is included in the definition of manifold to exclude examples like this. This example also shows that to avoid this kind of example, you must include some kind of global topological condition in the definition. It cannot be excluded by a purely local definition. *Note:* Theorem II.10.7 also has compactness as a hypothesis, and W is not compact. But, that is not the issue here. You can create a compact counterexample by starting with $X = Y = \mathbf{S}^1$ and doing a similar construction by letting A be the complement of a point in X.