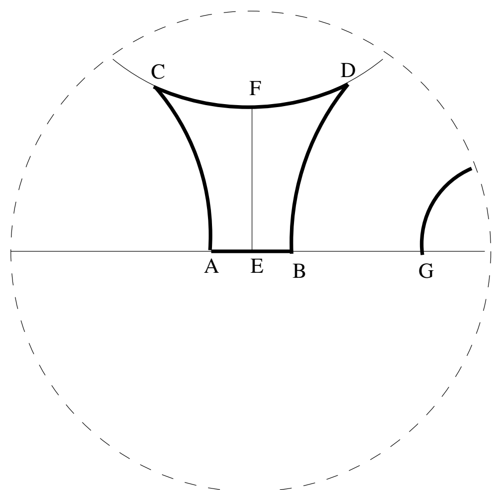
Math 4060 (Cherry) Theorem 34.5 (Saccheri)

As we discussed in class, the theorem of Saccheri (Theorem 34.5 in Hartshorne) is fundamental. The proof given by Hartshorne on page 309 is not quite correct, but the basic idea presented there can be fixed-up to give a correct proof. The problem comes in the argument surrounding the bottom figure. As I mentioned in class, by this part in the text, Hartshorne omits betweenness arguments that he feels the reader should be able to fill in on his or her own by this point in the course. However, it was unwise to omit one of the betweenness arguments in this proof, since that is ultimately the flaw in the argument given. Indeed, Hartshorne says: "Let the perpendicular to AB at G meet CD in H," but he does not give an argument for why the perpendicular to AB at G is not parallel to CD. Hartshorne has drawn his figure in the case that $\overline{EG} < \overline{EB}$. In this case, the perpendicular at G does meet $\overline{CD}$ at a point H as shown. This follows from the betweenness axiom B4 as follows. Consider $\triangle BED$. Since the perpendicular at G intersects side $\overline{EB}$ it must intersect either $\overline{ED}$ or $\overline{BD}$ by the betweenness axiom B4. However, it cannot intersect $\overline{BD}$ since $\overline{BD}$ is parallel to the perpendicular at G by Euclid I.27. Now, applying a similar argument to $\triangle DEF$, we find that the line perpendicular to AB at G intersects the segment $\overline{FD}$ at a point H as shown in Hartshorne's figure.

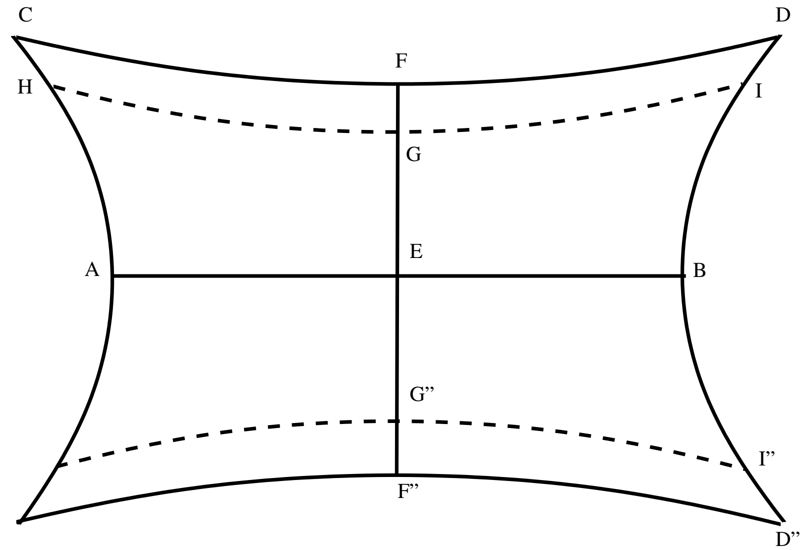
However, nothing says that $\overline{EG} < \overline{EB}$. In the event that $\overline{EG} > \overline{EB}$, there is no reason to expect the perpendicular at G not to be parallel to the line CD. Indeed, the following figure shows a Saccheri quadrilateral in the Poincaré plane (see section 39), where exactly this happens, and thus, Hartshorne's proof fails.



Corrected Proof of Theorem 34.5. Recall that the setting of the proof is that we have two Saccheri quadrilaterals $ABCD$ and $A'B'C'D'$ with midlines $\overline{EF}$ and $\overline{E'F'}$, respectively. The first case to consider is $\overline{EF} \cong \overline{E'F'}$. That case is handled correctly by the first half of Hartshorne's proof, so I will not repeat that part here.

If $\overline{EF} \ncong \overline{E'F'}$, then without loss of generality, assume that $\overline{E'F'} < \overline{EF}$. Thus, let G be the point in $\overline{EF}$ such that $\overline{EG} \cong \overline{E'F'}$ Erect a perpendicular at G (see the figure on the next page). By Euclid I.27, this is parallel to CD and AB. Hence, by applying the betweenness axiom B4 as above, the perpendicular at G intersects $\overline{AC}$ at a point H and $\overline{BD}$ at a point I. I now claim that $ABHI$ is a Saccheri quadrilateral. Indeed, by SAS, $\triangle AEG \cong \triangle BEG$. Hence, by ASA, $\triangle AGH \cong \triangle BGI$, and so $\overline{AH} \cong \overline{BI}$ as required for a Saccheri quadrilateral. Now, reflect the whole figure across the line AB. Now $FF''DD''$ and $GG''II''$ are both Saccheri quadrilaterals sharing the midline $\overline{EB}$. Hence if the angle at D is acute, then the angle at I is also acute, by the case proven at the start of Hartshorne's proof. But since $\overline{EG}$ is congruent to $\overline{E'F'}$, the angle at I being acute implies that the angle at D' is acute, again by the case of congruent midlines proven at the start of Hartshorne's proof.

We thus have that the Saccheri quadrilateral $A'B'C'D'$ has an acute angle if and only if the Saccheri quadrilateral $ABCD$ does, and similarly for obtuse angles and all right angles.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