Hilbert Axioms

Incidence Axioms

I1. Any two distinct points lie on a *unique* line.

I2. Every line contains at least two points.

I3. There exist three non-colinear points.

Parallel Axiom

P. Given a point P and a line ℓ, there is *at most one line* through P parallel to ℓ.

Betweenness Axioms

B1. If $A * B * C$, then A, B, and C are three distinct points all on a line and $C * B * A$.

B2. Given two distinct points A and B, there exists a point C so that $A * B * C$. [In other words, a line segment $\overline{AB}$ can always be extended to a line segment $\overline{AC}$ so that B is between A and C.]

B3. Given three distinct points A, B, and C all on a line, *exactly one of the following is true:*

$A * B * C$
$B * A * C$
$A * C * B$

[In other words, one and only one of the three points is in the middle.]

B4. Given three non-colinear points A, B, and C and a line ℓ not containing A, B, or C and containing a point D with $A * D * B$, then ℓ must also contain another point E such that exactly one of the following is true: $A * E * C$ or $B * E * C$. [In other words, a line going into a triangle and not through any of the vertices, must also come out through exactly one of the two remaining sides of the triangle.]

Segment Congruence Axioms

C1. Given a segment $\overline{AB}$ and a ray r with vertex C, there exists a *unique* point D on r with $\overline{CD} \cong \overline{AB}$.

C2. Congruence of segments is an equivalence relation.

C3. If $A * B * C$ and $D * E * F$ and $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, then $\overline{AC} \cong \overline{DF}$.

Angle Congruence Axioms

C4. Given an angle $\measuredangle BAC$, given a ray $\overrightarrow{DF}$, and given a side of line $\overleftrightarrow{DF}$, there exists a *unique* ray $\overrightarrow{DE}$ on the given side of line $\overleftrightarrow{DF}$, such that $\measuredangle BAC \cong \measuredangle EDF$.

C5. Congruence of angles is an equivalence relation.

C6. Side-angle-side.

Circle-Circle Intersection

E. Given two circles Γ and Δ, if Δ contains at least one point inside Γ and at least one point outside Γ, then Γ and Δ intersect.

Other Useful Stuff

Proposition 6.1. Two distinct lines intersect in *at most one* point.

Proposition 7.1 (plane seperation). Let ℓ be a line, and let A, B, and C be points not on ℓ.

(i) If A and B are on the same side of ℓ and B and C are on the same side of ℓ, then A and C are on the same side of ℓ.

(ii) If A and B are on the same side of ℓ and B and C are on opposite sides of ℓ, then A and C are on opposite sides of ℓ.

(iii) If A and B are on opposite sides of ℓ and B and C are on opposite sides of ℓ, then A and C are on the same side of ℓ.

Proposition 7.2 (line seperation). Let P be a point on a line ℓ. Let A, B, C be points on $\ell \setminus \{P\}$.

(i) If A and B are on the same side of P and B and C are on the same side of P, then A and C are on the same side of P.

(ii) If A and B are on the same side of P and B and C are on opposite sides of P, then A and C are on opposite sides of P.

(iii) If A and B are on opposite sides of P and B and C are on opposite sides of P, then A and C are on the same side of P.

Proposition 7.3 (crossbar theorem). Given an angle $\measuredangle BAC$ and a point D on the inside of $\measuredangle BAC$, then $\overrightarrow{AD}$ intersects $\overline{BC}$.

Exercise 7.1. (i) If $A * B * C$ and $B * C * D$, then $A * B * D$ and $A * C * D$. (ii) If $A * B * D$ and $B * C * D$, then $A * B * C$ and $A * C * D$.

Proposition 8.3 (subtraction of segments). Given points A, B, C, D, E, and F with $B \in \overline{AC}$ and $F \in \overrightarrow{DE}$ and $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$, then $E \in \overline{DF}$ and $\overline{EF} \cong \overline{BC}$.

Proposition 8.4 (Inequalities of segments). Given two segments $\overline{AB}$ and $\overline{CD}$, *exactly one of* the following is true: (i) $\overline{AB} \cong \overline{CD}$, (ii) $\overline{AB} < \overline{CD}$, (iii) $\overline{CD} < \overline{AB}$.

Proposition 9.4 (addition of angles). Suppose $\measuredangle BAC$ is an angle, and ray $\overrightarrow{AD}$ is in the interior of $\measuredangle BAC$. Suppose $\measuredangle D'A'C \cong \measuredangle DAC$, $\measuredangle B'A'D' \cong \measuredangle BAD$, and the rays $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ are on opposite sides of the line $A'D'$. Then, the rays $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ form an angle with $\measuredangle B'A'C' \cong \measuredangle BAC$, and the ray $\overrightarrow{A'D'}$ is in the interior of $\measuredangle B'A'C'$.

Proposition 9.5 (Inequalities of angles). (a) If $\alpha \cong \alpha'$ and $\beta \cong \beta'$, then $\alpha < \beta$ if and only if $\alpha' < \beta'$. (b) If $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$. (c) Given any two anlges α and β, exactly one of the following holds: $\alpha < \beta$, $\alpha \cong \beta$, or $\beta < \alpha$.