Math 4060 (Cherry)

Outline of a Proof of Axioms C4–C6 for \( \mathbb{R}^2 \).

This is not a homework assignment or an extra credit opportunity. Rather, this is a set of exercises to help those interested in doing so to work through a proof of axioms C4–C6 for \( \mathbb{R}^2 \) based on linear algebra. You are not being asked to turn these exercises in. They are only for those interested in working through them on their own. I am happy to discuss any of these exercises during office hours.

Hartshorne postpones the proof that \( \mathbb{R}^2 \) satisfies axioms C4–C6 because he wants to prove this result in somewhat greater generality and make a connection to the group of rigid motions. If we only want to work with \( \mathbb{R}^2 \), here is a way to do it based on two dimensional linear algebra.

Let \( \vec{u} = (u_1, u_2) \) and \( \vec{v} = (v_1, v_2) \). Recall that the standard inner product (also called dot product or scalar product) is defined by \( \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 \). This is also often written in the notation: \( < \vec{u}, \vec{v} > = u_1 v_1 + u_2 v_2 \).

(a) Prove the Cauchy-Schwarz Inequality: \( |\vec{u} \cdot \vec{v}| \leq |\vec{u}| |\vec{v}| \), and show equality holds if and only if \( \vec{u} \) and \( \vec{v} \) are linearly dependent, meaning \( \vec{v} = \vec{0} \) or \( \vec{u} = t \vec{v} \) for some real number \( t \).

Hint: The case that \( \vec{v} = \vec{0} \) is easy. If \( \vec{v} \neq \vec{0} \), then consider for all real numbers \( t \) that

\[
0 \leq (\vec{u} + t \vec{v}) \cdot (\vec{u} + t \vec{v}).
\]

Multiply out the right hand side to get a quadratic equation \( at^2 + bt + c \) with \( a > 0 \). Note you should have \( a, b \) and \( c \) in terms of dot products of \( \vec{u} \) and \( \vec{v} \) without any \( t \)'s. Use the fact that \( at^2 + bt + c \geq 0 \) for all real \( t \) and \( a > 0 \) to conclude \( b^2 - 4ac \leq 0 \), and hence the Cauchy-Schwarz inequality.

Given two non-zero vectors \( \vec{u} \) and \( \vec{v} \), define

\[
\cos(\vec{u}, \vec{v}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}.
\]

I have chosen the name \( \cos \) because this defines the cosine of the angle between the vectors, but we do not need to know anything about cosine or trigonometry to do this. Simply take this as a definition. Note that Cauchy-Schwarz implies \( |\cos(\vec{u}, \vec{v})| \leq 1 \).

(b) Show that for positive real numbers \( s \) and \( t \), we have \( \cos(s \vec{u}, t \vec{v}) = \cos(\vec{u}, \vec{v}) \).

Given a point \( A \) in the plane with coordinates \( (x, y) \), let \( \vec{A} \) denote the vector \( (x, y) \), i.e., the vector pointing from the origin to the point \( A \). Suppose \( \angle BAC = \angle B'AC' \) – not congruent!, actually equal meaning \( B' \) is on the ray \( AB \) and \( C' \) is on the ray \( AC \). Then, show

\[
\cos(\vec{B} - \vec{A}, \vec{C} - \vec{A}) = \cos(\vec{B'} - \vec{A}, \vec{C'} - \vec{A})
\]

and hence \( \cos(\vec{B} - \vec{A}, \vec{C} - \vec{A}) \) really only depends on the the angle \( \angle BAC \) in Hartshorne’s sense, and not the choice of individual points \( A, B, \) and \( C \) representing the angle. Thus, it makes sense to define

\[
\angle BAC \cong \angle EDF \quad \text{if and only if} \quad \cos(\vec{B} - \vec{A}, \vec{C} - \vec{A}) = \cos(\vec{E} - \vec{D}, \vec{F} - \vec{D}).
\]

Note that defining this in terms of “equals” automatically gives us a notion of congruence that is an equivalence relation and hence satisfies C5.

(c) Show that if \( \angle BAC \) is an angle in the sense of Hartshorne, so \( A, B, \) and \( C \) are NOT colinear, then

\[
|\cos(\vec{B} - \vec{A}, \vec{C} - \vec{A})| < 1.
\]

Again, do this directly from the definition in terms of dot products, not using anything you know from trigonometry about cosines. In other words, show that if you get

\[
|\cos(\vec{B} - \vec{A}, \vec{C} - \vec{A})| = 1,
\]

then \( A, B, \) and \( C \) must be colinear. Hint: Use part (a).

Continued on the back
(d) Check axiom C4 as follows. You are given an angle $\angle BAC$. By axiom C1, you may assume that
\[ |\vec{B} - \vec{A}| = |\vec{C} - \vec{A}| = 1, \]
and this will make the algebra easier. You are also given a segment $\overline{DE}$, and you may again assume $|\vec{E} - \vec{D}| = 1$. You need to show you can find exactly two points $F$ (on each side of the line containing $D$ and $E$) so that
\[ \cos(\vec{F} - \vec{D}, \vec{E} - \vec{D}) = \cos(\vec{B} - \vec{A}, \vec{C} - \vec{A}). \]
What this amounts to is the following. Suppose $\vec{E} - \vec{D} = (a, b)$ with $a^2 + b^2 = 1$, where we have this last equation because we assumed $|\vec{E} - \vec{D}| = 1$. Let $\alpha = \cos(\vec{B} - \vec{A}, \vec{C} - \vec{A})$, and remember we have $|\alpha| < 1$. Now, show that the system of equations
\[
\begin{align*}
    as + bt &= \alpha \\
    s^2 + t^2 &= 1
\end{align*}
\]
has exactly two solutions. Show the points $\vec{F} = \vec{D} + (s, t)$ are exactly the points needed to get C4.

Axiom C6, or SAS, will follow if you can show given an angle $\angle BAC$ you can compute everything in terms of $c = |\vec{B} - \vec{A}|$, $b = |\vec{C} - \vec{A}|$, and $\alpha = \cos(\vec{B} - \vec{A}, \vec{C} - \vec{A})$.

(e) First show that
\[ |\vec{C} - \vec{B}|^2 = (\vec{C} - \vec{B}) \cdot (\vec{C} - \vec{B}) = b^2 + c^2 - 2bc\alpha. \]
Note this is an “analytic geometry” proof of the “Law of Cosines” from precalculus. One way to do it is note that
\[ \vec{C} - \vec{B} = (\vec{C} - \vec{A}) - (\vec{B} - \vec{A}) \]
and then compute using rules of dot products.

Because we now have all the side lengths in terms of $b$, $c$, and $\alpha$, all we need to show is that we can express the remaining dot products in terms of these quantities. Indeed,
\[
\begin{align*}
    (\vec{C} - \vec{B}) \cdot (\vec{A} - \vec{B}) &= -(\vec{C} - \vec{B}) \cdot (\vec{B} - \vec{A}) \\
    &= -(\vec{C} - \vec{A} - (\vec{B} - \vec{A}) \cdot (\vec{B} - \vec{A}) \\
    &= -(\vec{C} - \vec{A}) \cdot (\vec{B} - \vec{A}) + (\vec{B} - \vec{A}) \cdot (\vec{B} - \vec{A}) \\
    &= c^2 - bca. \\
\end{align*}
\]
Similarly,
\[ (\vec{A} - \vec{C}) \cdot (\vec{B} - \vec{C}) = b^2 - bca. \]
These two dot products determine the remaining angles.