Eigenvalues, Eigenvectors, and Differential Equations

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The concepts of eigenvalue and eigenvector occur throughout advanced mathematics. They are often introduced in an introductory linear algebra class, and when introduced there alone, it is hard to appreciate their importance. The most likely other place one will meet eigenvalues in the undergraduate curriculum is in a course in differential equations, although the presence of eigenvalues may be hidden under the surface and not mentioned explicitly. The purpose of these notes is to give a quick introduction to the significance of eigenvalues and eigenvectors in the study of differential equations. Only the simplest cases are treated here, and a more thorough study is left for a proper course on differential equations.

1 A differential equation from calculus.

Let $y = f(t)$ and suppose that y satisfies the differential equation:

$$f'(t) = \lambda f(t) \text{ which can also be written } \frac{dy}{dt} = \lambda y,$$

where λ is a constant. This equation says that the rate of growth of y is proportional to y, and λ is the proportionality constant. You may remember that such equations are associated to population growth (the rate at which a bacteria culture reproduces is proportional to the amount of bacteria already present) and radioactive decay (the rate at which a substance decays is proportional to the amount of substance there). We have chosen to call our constant λ to foreshadow the fact that it will be viewed as an eigenvalue. The above equation is called a "differential equation" because it is an equation that involves a "derivative." It is called an "ordinary differential equation" because it only involves "ordinary" derivatives, and not "partial derivatives" from multivariable calculus. This equation is a "first order" equation because it only involves a first derivative. It is also called a "linear" equation, since y and dy/dt only appear as linear terms, and hence the eventual connection with linear algebra. Finally, the equation is called "autonomous", because it does not change with time t (although its solution does). Solving a differential equation means finding all possible functions $f(t)$ that satisfy the equation. Unlike most differential equations, it happens that the above differential equation is easy to solve, and you may have learned how to solve it in your freshman calculus class. Namely, the variables can be "separated" so that the left side contains only y and the right side contains only t :

$$\frac{dy}{y} = \lambda dt.$$

We can then integrate both sides to get

$$\int \frac{dy}{y} = \int \lambda dt$$

and by actually finding the antiderivatives

$$\ln y = \lambda t + c,$$

where c is a constant of integration. We do not need a constant of integration on each side, because the constant on the left can be combined with the constant on the right. Now, exponentiating both sides of the equation, we find

$$y = e^{\lambda t + c} = e^c \cdot e^{\lambda t}.$$

Because e^c is also a constant, we can denote it simply as C, and we have

$$y = Ce^{\lambda t}.$$

This is known as the "general solution" of the differential equation. Note that when $t = 0$, then $y = C$, and so it is also common to denote C by y_0 and write

$$y = y_0 e^{\lambda t}.$$

If we specify a value of y_0, then we have what is called a "particular solution" to the equation, and it is the solution of the differential equation that satisfies the "initial condition" $y(0) = y_0$.

2 Systems of differential equations.

The easiest way to make a connection to linear algebra is to consider systems of differential equations. Your textbook discusses examples coming from simple electric circuits, but I find it more fun to consider some "Romeo and Juliet" examples. I learned these examples as a technique for teaching the role of eigenvalues in differential equations from the book:

Steven H. Strogatz, *Nonlinear Dynamics and Chaos,* Addison Wesley, 1994.

The situation is as follows. Let R or $R(t)$ denote Romeo's affection for Juliet at time t. We will say $R > 0$ corresponds to positive affection *i.e.,* love, for Juliet, and $R < 0$ corresponds to negative affection, *i.e.,* hate. Similarly, J will denote Juliet's affection for Romeo at time t. Then, dR/dt and dJ/dt represent how Romeo and Juliet's affections for each other are changing at an instant in time.

To begin, let's consider a case where Romeo and Juliet's feelings for each other depend only on themselves and not on how the other feels about them. More precisely, suppose the change in Romeo's affection for Juliet is proportional to how much affection he already has for her, and similarly for Juliet. In equations, this could be written:

$$\begin{aligned} \frac{dR}{dt} &= \lambda_1 R \\ \frac{dJ}{dt} &= \lambda_2 J \end{aligned}$$

Although this is a system of equations, the two equations are completely independent of one another. Thus, they can be solved as above to get

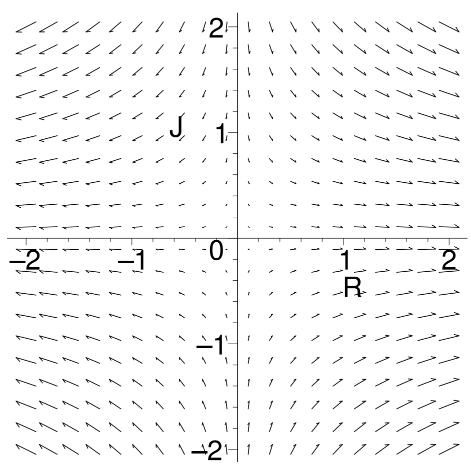
$$\begin{aligned} R &= R_0 e^{\lambda_1 t} \\ J &= J_0 e^{\lambda_2 t}. \end{aligned}$$

However, to emphasize the connection with linear algebra, let's write the original system in matrix form:

$$\begin{bmatrix} dR/dt \\ dJ/dt \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}.$$

The fact that the matrix is diagonal is what makes the equations so easy to solve. If $\lambda_1 > 0$ and Romeo starts out with some love for Juliet ($R_0 > 0$), then Romeo's love for Juliet feeds off itself and grows exponentially. If $R_0 < 0$, then Romeo's hate for Juliet feeds off itself and grows exponentially. On the other hand, if $\lambda_1 < 0$, then R decays exponentially, so Romeo's passion (either positive or negative) for Juliet gradually whithers away. Juliet's feelings are similarly determined by λ_2.

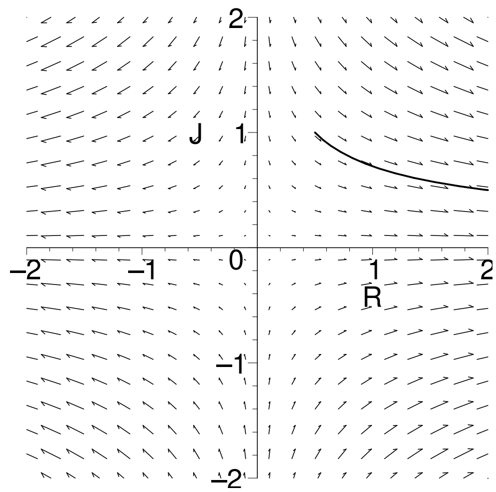
Although in this case, it was not hard to find an exact formula for Romeo and Juliet's feelings as a function of time, often we are not so much interested in such exact formulas and only want to know what happens, *i.e.,* do Romeo and Juliet fall and eventually remain in love with each other? This kind of question can often be answered by drawing what is known as a "phase diagram" or "phase portrait." To plot a phase diagram for our Romeo and Juliet equations, at each point (R, J) in the plane, we plot the vector $\begin{bmatrix} dR/dt \\ dJ/dt \end{bmatrix}$. By way of example, let's take $\lambda_1 = 2$ and $\lambda_2 = -1$. The corresponding phase plane would then look like:



Notice, for example, that at the point $(1, 1)$, the vector

$$\begin{bmatrix} dR/dt \\ dJ/dt \end{bmatrix} = \begin{bmatrix} 2R \\ -J \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

has been plotted. In fact, the vectors have been shortened to make the plot more readable because it is mainly the direction of the vector we are interested in and the relative size of the vector compared with its neighboring vectors. The actual size of the vector is not important. The arrows show how Romeo and Juliet's affections will change. We get a solution trajectory by starting at a given initial point and then following the arrows. For example, suppose that we start at the point $(R, J) = (0.5, 1)$, corresponding to Romeo having a slight interest in Juliet and Juliet having a moderate interest in Romeo. Following the arrows, we find the solution trajectory drawn here:



Thus, we see that Juliet's love for Romeo decays over time and Romeo's love for Juliet grows stronger. We can see that eventually Romeo will be head over heels in love with Juliet, but Juliet will eventually be indifferent toward Romeo.

As the matrix corresponding to this system was diagonal, the two eigendirections are the coordinate axes. Thus, we see that no matter where we start, because Juliet's affection decays, that we end up approaching the eigendirection corresponding to the R-axis as an asymptote. Of course if we start on the R-axis, we stay there. The J-axis is also an eigenspace, and if we start on the J-axis, we stay there, but approaching the origin. If we start near the J-axis, we eventually move away from it. When this happens the J-axis is called a "repeller." Also, we see that eventually we approach the R-axis, so we call the R-axis an "attractor."

2.1 A coupled system.

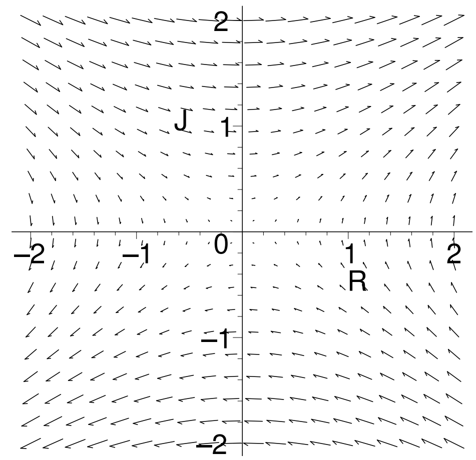
Now let's look at a more complicated situation. This time, instead of Romeo and Juliet's feelings feeding off their own feelings for each other, suppose that their feelings change based on how the other feels toward them. In other words, suppose

$$\frac{dR}{dt} = aJ$$
$$\frac{dJ}{dt} = bR$$

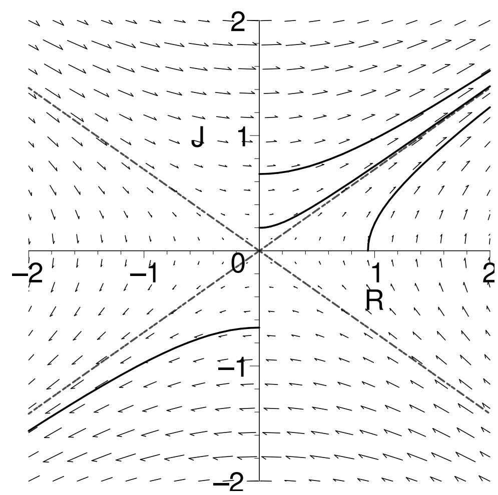
with a and b positive constants. Thus, Romeo grows fonder of Juliet the more she likes him, and Juliet grows fonder of Romeo the more he likes her. This system can be written in matrix notation as

$$\begin{bmatrix} dR/dt \\ dJ/dt \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}.$$

We will begin by plotting the phase diagram for this system



If we add in plots for a few solution trajectories (solid curve), we see as before a repeller and attractor (dotted lines).



We will show the repeller and attractor are the eigendirections of the matrix.

To find the eigenvalues of the matrix

$$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$$

we compute the determinant

$$\begin{vmatrix} -\lambda & a \\ b & -\lambda \end{vmatrix}$$

and set it equal to zero to get the characteristic equation

$$\lambda^2 - ab = 0,$$

so the two eigenvalues are

$$\lambda = \pm\sqrt{ab}.$$

We find that

$$\begin{bmatrix} \sqrt{a/b} \\ 1 \end{bmatrix}$$

is an eigenvector with eigenvalue $\sqrt{ab}$ and

$$\begin{bmatrix} -\sqrt{a/b} \\ 1 \end{bmatrix}$$

is an eigenvector with eigenvalue $-\sqrt{ab}$. These vectors point in the direction of the repeller and attractor. Thus, we see that depending on where we start, either Romeo and Juliet both end up more and more in love or they end up hating each other more and more. Who has the stronger feelings is determined by whether $a > b$ or $b > a$. The pictures above show the case $a > b$.

In fact, we can also use eigenvalues and eigenvectors to solve for R and J in terms of t. Indeed, consider the change of basis matrix

$$P = \begin{bmatrix} \sqrt{a/b} & -\sqrt{a/b} \\ 1 & 1 \end{bmatrix} \quad \text{which has inverse} \quad P^{-1} = \begin{bmatrix} \frac{1}{2}\sqrt{b/a} & \frac{1}{2} \\ -\frac{1}{2}\sqrt{b/a} & \frac{1}{2} \end{bmatrix}.$$

Then,

$$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} = P \begin{bmatrix} \sqrt{ab} & 0 \\ 0 & -\sqrt{ab} \end{bmatrix} P^{-1},$$

and we therefore have

$$\begin{bmatrix} dR/dt \\ dJ/dt \end{bmatrix} = P \begin{bmatrix} \sqrt{ab} & 0 \\ 0 & -\sqrt{ab} \end{bmatrix} P^{-1} \begin{bmatrix} R \\ J \end{bmatrix}.$$

If we set

$$\begin{bmatrix} x \\ y \end{bmatrix} = P^{-1} \begin{bmatrix} R \\ J \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = P^{-1} \begin{bmatrix} dR/dt \\ dJ/dt \end{bmatrix},$$

and so

$$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} \sqrt{ab} & 0 \\ 0 & -\sqrt{ab} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Because this is a diagonal system, we know

$$\begin{aligned} x &= x_0 e^{\sqrt{ab}t} \\ y &= y_0 e^{-\sqrt{ab}t} \end{aligned}$$

To switch back to R and J, we use

$$\begin{bmatrix} R \\ J \end{bmatrix} = P \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sqrt{a/b} & -\sqrt{a/b} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_0 e^{\sqrt{ab}t} \\ y_0 e^{-\sqrt{ab}t} \end{bmatrix} = \begin{bmatrix} \sqrt{a/b}(x_0 e^{\sqrt{ab}t} - y_0 e^{-\sqrt{ab}t}) \\ x_0 e^{\sqrt{ab}t} + y_0 e^{-\sqrt{ab}t} \end{bmatrix}.$$

We will not be particularly interested in this explicit formula. I only want to point out that by using a change of basis matrix associated to a basis of eigenvectors, we can find an exact formula if we want, and that the the eigenvalues appear in these formulas.

2.2 Two equally cautious lovers.

Let us now consider the system

$$\begin{bmatrix} dR/dt \\ dJ/dt \end{bmatrix} = \begin{bmatrix} -a & b \\ b & -a \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix},$$

where $a > 0$ and $b > 0$. In this circumstance, Romeo and Juliet are both affraid of their own feelings toward the other, so the more Romeo loves Julliet, the more he pulls back, and similarly for Juliet. On the other hand, they respond positively to the other's affection for them, so the more Juliet likes Romeo, the more he tends to like her. If we find the characteristic equation for this system, we find

$$\begin{vmatrix} -a-\lambda & b \\ b & -a-\lambda \end{vmatrix} = 0,$$

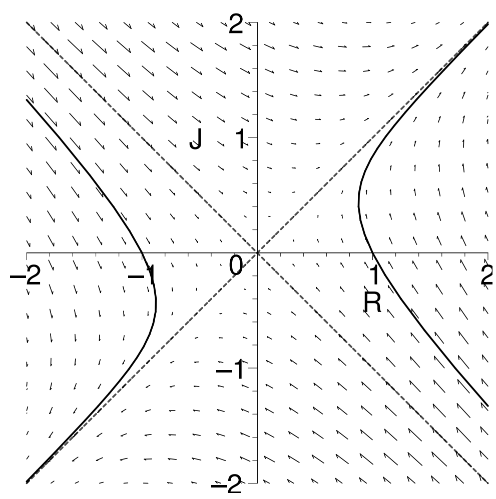
which leads to

$$(-a-\lambda)^2 - b^2 = 0,$$

or in other words,

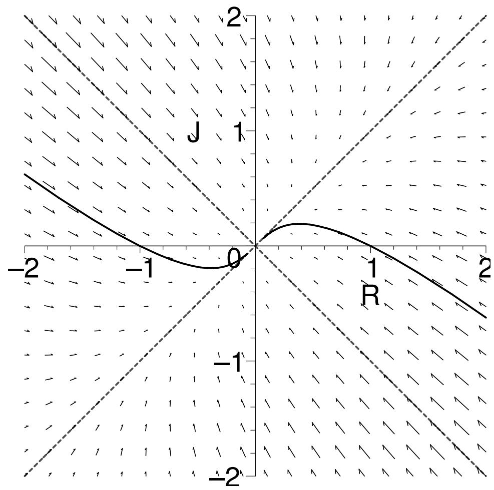
$$\lambda = -a \pm b.$$

We also find that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $-a+b$ and that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector for the eigenvalue $-a-b$. In order to analyze what happens, we need to now consider two cases. First consider the case $b > a$, which corresponds to Romeo and Juliet being more sensitive to each other's feelings than to their own. In this case, the eigenvalue $-a+b$ is positive, and of course $-a-b$ is negative. In this case the eigendirection $R = J$ will be an attractor and the tranjectories will move out away from the origin so that either Romeo and Juliet both end up loving each other or both end up hating each other.



There is one exceptional case. Namely, if we happen to start on the line $R = -J$, then both Romeo and Juliet's feelings fade to indifference over time, and the trajectory ends up at the origin.

In the case when $a > b$, meaning that Romeo and Juliet are both more sensitive to their own feelings than to each others, we find that both eigenvalues are negative, so both Romeo and Juliet's feelings decay toward mutual indifference, no matter where they start. Because the eigenvalue in the direction $R = -J$ is more negative, indicating faster decay in that direction, the eigendirection $R = J$ remains an attractor.

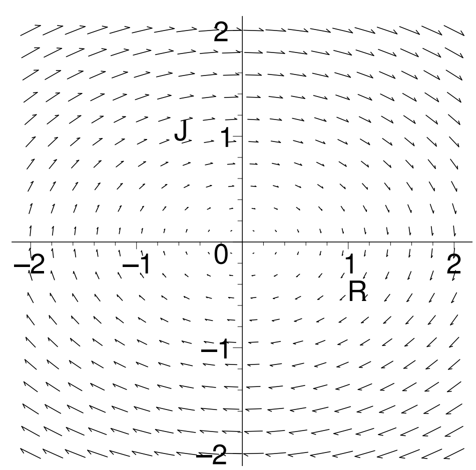


2.3 An example with imaginary eigenvalues.

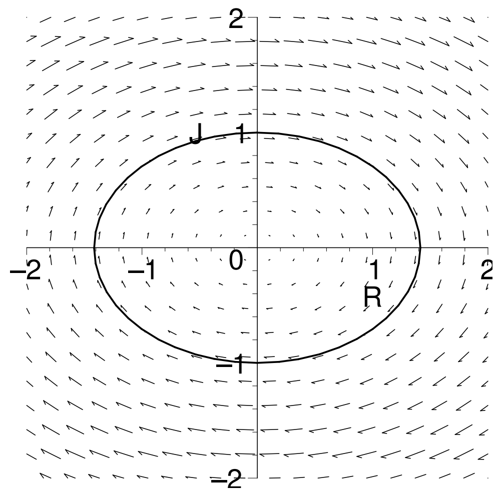
Now consider the system of equations

$$\frac{dR}{dt} = aJ$$
$$\frac{dJ}{dt} = -bR,$$

with a and b positive. In this circumstance, Romeo responds positively to Juliet's affection for him, but Juliet likes Romeo more when Romeo dislikes her, and conversely, she likes Romeo less when Romeo likes her more. The phase diagram for this system is



In this case, we see that a typical solution trajectory looks like:



Suppose that things started with Juliet interested in Romeo, but Romeo ambivalent to Juliet. Then Juliet's interest in Romeo makes Romeo more fond of Juliet. But, this then turns Juliet off, and she becomes less enthralled with Romeo, until she begins to dislike him. As Juliet's dislike for Romeo grows, Romeo loses his affection for Juliet, until finally he also grows to dislike her. But, this causes Juliet to become more attracted to Romeo, until finally Juliet becomes fond of Romeo again, and the cycle repeats itself.

Writing the system that led to this phase portrait in matrix form, we get

$$\begin{bmatrix} dR/dt \\ dJ/dt \end{bmatrix} = \begin{bmatrix} 0 & a \\ -b & 0 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}.$$

This time we find that the characteristic equation is $\lambda^2 = \sqrt{-ab}$, and so the eigenvalues are pure imaginary: $\lambda = \pm i\sqrt{ab}$. Since the eigenvalues are imaginary, we have no real eigenvectors, and so we do not see any asymptotic directions associated to eigendirections in our phase diagram.

If we allow ourselves to use complex eigenvectors to diagonalize our matrix, we will find the explicit formulas for the solution to this system of equation will involve $e^{i\sqrt{ab}t}$ and $e^{-i\sqrt{ab}t}$. Recalling that

$$e^{i\sqrt{ab}t} = \cos(\sqrt{ab}t) + i\sin(\sqrt{ab}t) \quad \text{and} \quad e^{-i\sqrt{ab}t} = \cos(\sqrt{ab}t) - i\sin(\sqrt{ab}t),$$

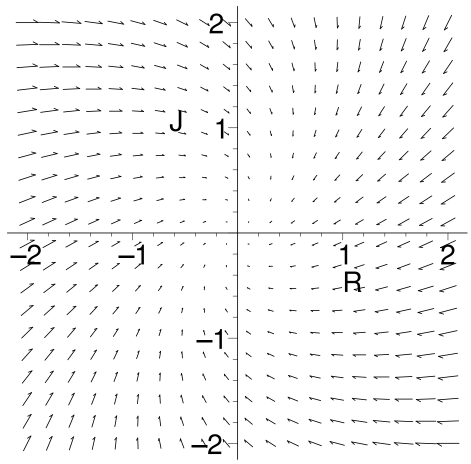
we can re-write our solution in terms of $\cos(\sqrt{ab}t)$ and $\sin(\sqrt{ab}t)$. Notice that the period of the cycle is directly related to the eigenvalues.

2.4 A spiral.

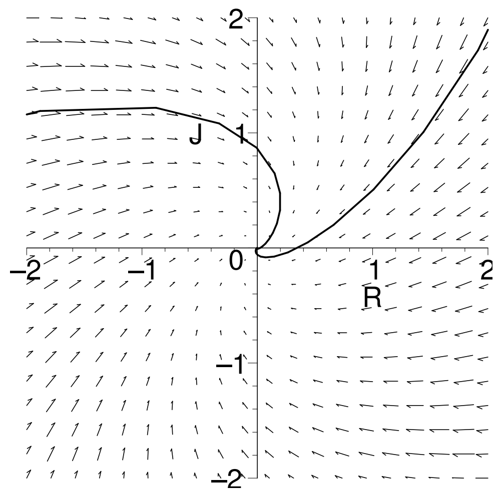
To conclude these notes, consider the system

$$\begin{bmatrix} dR/dt \\ dJ/dt \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}.$$

The phase portrait for this system looks like:



If we draw in a couple of solution trajectories, we see



In this case the eigenvalues are complex and contain both a non-zero real and imaginary part:

$$\lambda = -\frac{3}{2} \pm i\frac{\sqrt{3}}{2}.$$

The fact that the real part is negative corresponds to exponential decay. The imaginary part is related to the period of rotation. If the real part had been positive, the spirals would move outward as time advances. If we were to use complex eigenvectors to diagonalize our matrix, we would find our explicit solution to be built out of

$$e^{-3t/2}\cos(t\sqrt{3}/2) \qquad \text{and} \qquad e^{-3t/2}\sin(t\sqrt{3}/2),$$

and again we see the eigenvalue show up in the explicit formula, the real part of the eigenvalue affecting the decay rate and the imaginary part of the eigenvalue affecting the period.

The theme here is that the eigenvalues tell you the qualitative behavior of the system. When the eigenvalues are real, some of the eigenvectors point in asymptotic directions. Other eigenvectors point in the direction that repels solutions away. There are many subtleties involved in finding explicit formulas as solutions that have not been touched on here. These issues are more properly discussed in a differential equations class. In particular, I have not addressed here what to do in the case that there is a repeated eigenvalue and the matrix cannot be diagonalized.