**Math 1400/1650 (Cherry): Quadratic Functions**

A **quadratic function** is a function $Q(x)$ of the form

$$\boxed{Q(x) = ax^2 + bx + c \qquad \text{with } a \neq 0.}$$

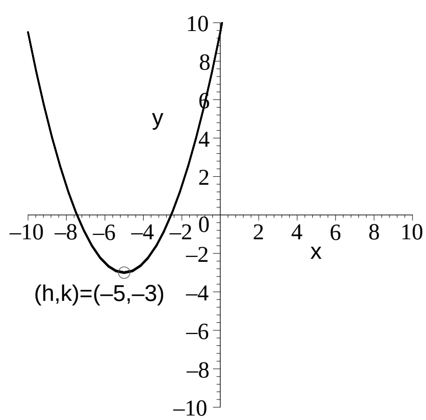
For example, $Q(x) = 3x^2 - 7x + 5$ is a quadratic function, and here $a = 3$, $b = -7$ and $c = 5$. Although $ax^2+bx+c$ is probably the form we are most familiar with from high school for quadratic functions, this is **not what we refer to as standard form. Standard form** for a quadratic function means write the quadratic function in the form:

$$\textbf{Standard From: } \boxed{Q(x) = a(x-h)^2 + k.}$$

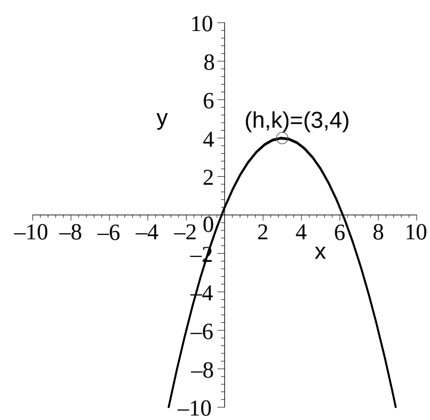
**Tip!** The "$a$" value in standard form is the same "$a$" in the form $ax^2 + bx + c$.

**Why standard form is useful.** Standard form is the most useful form for a quadratic function because it tells you the most important information about the function. Namely, $(h, k)$ is the **vertex** of the graph of $Q(x)$. If $a > 0$ the graph opens up; if $a < 0$ the graph opens down:

Graph of $y = a(x-h)^2 + k$ with $a > 0$




Graph of $y = a(x-h)^2 + k$ with $a < 0$




This hand-out will explain several methods for putting a quadratic function in standard form. You do not need to learn all the methods. But, everyone should learn at least one algebraic method. Different people find different methods easier to use.

**Algebraic method #1 (completing the square).** This method is perhaps the fastest when the numbers are nice. People who do not memorize formulas well, but who are good at doing algebra in their head tend to like this method. If you are not good at algebra in your head, you should probably concentrate on the other methods. Let us put the quadratic function $Q(x) = 3x^2 + 36x + 1$ in standard form.

> **Step 0:** Write the function in decreasing powers of $x$: $Q(x) = 3x^2 + 36x + 1$

> **Step 1:** Factor out the "$a$" from the first two terms: $Q(x) = 3(x^2 + 12x) + 1$.

The tricky step is to try to make $x^2 + 12x$ look like a perfect square. That is, we want to find something so

$$(x + \Box)^2 = x^2 + 12x + \ldots$$

Note that

$$(x + \Box)^2 = x^2 + 2\Box x + \Box^2.$$

Thus, we divide $12$ by $2$ to get $6$ and consider $(x + 6)^2 = x^2 + 12x + 36$. Notice this matches $x^2 + 12x$ except for the constant term.

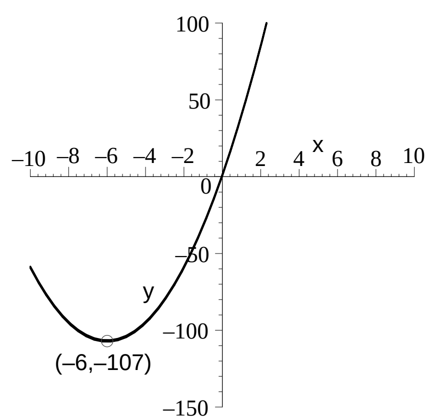
> **Step 3:** perfect square: $Q(x) = 3(x^2 + 12x + 36 - 36) + 1$.

Notice we have added and subtracted $36$ so we haven't changed anything.

> **Step 4:** Factor the perfect square: $Q(x) = 3((x + 6)^2 - 36) + 1$.

> **Step 5:** Combine the constant terms: $Q(x) = 3(x+6)^2 - 3 \cdot 36 + 1 = 3(x+6)^2 - 107$.

Thus $Q(x) = 3(x + 6)^2 - 107$ is standard form, and we see immediately that the vertex is $(-6, -107)$. Because $a = 3 > 0$, we know the parabola opens up, and so the minimum value of $Q$ is $-107$ and it occurs when $x = -6$.



**Algebraic method #2 (work backwards).** This method is good for people who do not memorize well and find the previous method difficult to do. The idea here is to multiply out standard form. Let's take the same example.

**Step 0:** Write the function in decreasing powers of $x$: $Q(x) = 3x^2 + 36x + 1$

**Step 1:** Set equal to standard form: $3x^2 + 36x + 1 = 3(x - h)^2 + k$.

Note that we have made use of the fact that the $a$ values are the same in the form we started with and in standard form. That is, we plugged in $a = 3$ on the right hand side.

**Step 2:** Multiply out the right-hand side: $3x^2 + 36x + 1 = 3(x^2 - 2hx + h^2) + k$.

**Step 3:** Continue to multiply out on the right: $3x^2 + 36x + 1 = 3x^2 - 6hx + 3h^2 + k$.

**Step 4:** Cancel the $x^2$ terms: $36x + 1 = -6hx + 3h^2 + k$.

Now we compare what is in front of "$x$" on the left and right. On the left we have $36$ and on the right we have $-6h$. Thus, we must have that $36 = -6h$, which we can use to solve for $h$.

**Step 5:** solve for $h$: $h = -6$.

Now we compare the constant terms on the right and left. We have $1$ on the left and $3h^2 + k$ on the right. Hence $1 = 3h^2 + k$.

**Step 6:** plug in for $h$: $1 = 3(-6)^2 + k$.

**Step 7:** solve for $k$: $k = 1 - 3(-6)^2 = -107$.

**Step 8:** plug $h$ and $k$ into standard form: $Q(x) = 3(x-(-6))^2-107 = 3(x+6)^2-107$.

**Algebraic method #3 (memorize a formula).** This is the method for you if you are good at memorizing formulas. If you can memorize the formula, and especially if the numbers are bad and you have a calculator to help you multiply, then this is certainly the easiest way that will always work.

**Step $-1$:** memorize the formula $\boxed{h = -\frac{b}{2a}.}$

**Step 0:** Write the function in decreasing powers of $x$: $Q(x) = 3x^2 + 36x + 1$

**Step 1:** Use the formula: $h = -\frac{b}{2a} = -\frac{36}{2 \cdot 3} = -\frac{36}{6} = -6$.

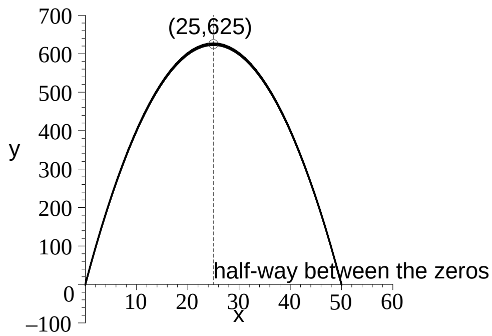
**Step 2:** Use $k = Q(h)$: $k = Q(-6) = 3(-6)^2 + 36(-6) + 1 = -107$.

**Step 3:** plug $h$ and $k$ into standard form: $Q(x) = 3(x-(-6))^2-107 = 3(x+6)^2-107$.

**Algebraic method #4 (factored short-cut). Warning!** This method does not work for all quadratic functions. It is a short-cut that lets you avoid multiplying out if for some reason your quadratic function is factored. It is worth learning this short-cut because in many word problems, the quadratic function naturally comes factored. For example, suppose that a rectangle has perimeter 100. Let $x$ be the length of the rectangle. Find the area $A$ as a function of $x$. Well, we have that

$$A(x) = \text{length} \cdot \text{width} = x(50 - x).$$

Note that that the $50 - x$ comes from the fact that the perimeter is $100$, so length$+$width $= 50$. This function naturally comes factored. To use any of the methods above, we would need to first multiply this function out. This is not so hard, but it is an extra step. What we can do instead is use a special fact about quadratic functions: *if a quadratic function has zeros, then the vertex is exactly half-way between the zeros.* In this case, we see that $A(0) = 0$ and $A(50) = 0$. Half-way between $0$ and $50$ is $25$, so $h = 25$. To figure out $k$, we use $k = A(h) = A(25) = 25(50 - 25) = 25 \cdot 25 = 625$. Now $a$ is $-1$, so $A(x) = -(x - 25)^2 + 625$ is standard form.



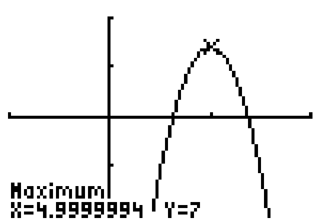

Notice that standard form tells us a lot. It tells us that the most area we can have with a rectangle with a perimeter of 100 is 625, and it tells us that to get that much area, we need to make its length be 25.

**Graphing Calculator Method.** You can also convert to standard form with the help of a calculator. Plug the formula in using `Y=`. Graph in a window where you can see the vertex. On the calc menu, if the parabola opens up, choose "minimum" and if the parabola opens down, choose "maximum." This will find the values of $h$ and $k$ for you. Then simply plug into the equation

$$Q(x) = a(x - h)^2 + k.$$

Remember the minus sign before $h$.

For example, if we look at $-2x^2 + 20x - 43$ on a graphing calculator and find the maximum (because $a$ is negative), we see




that the vertex is at $(5, 7)$ and so standard form of the equation is: $\boxed{-2(x - 5)^2 + 7.}$

**Finding roots (or zeros) of a quadratic function**

**Algebraic Method #1 (factoring):** Suppose we want to solve $x^2 - x - 6 = 0$. One approach is to factor the quadratic function as $(x-3)(x+2) = 0$. Then we observe that we can only multiply two numbers together and get zero if at least one is zero. So we know $x - 3 = 0$ or $x + 2 = 0$. Solving each of these equations, we find $x = 3$ and $x = -2$ as our roots or zeros. Of course this method is the best method to use if the numbers are easy or the function is already factored.

**Algebraic Method #2 (standard form):** Finding roots is easy if you put the quadratic function in standard form, or if it is already in standard form. For example, suppose we want to solve

$$2(x-5)^2 - 50 = 0.$$

Simply move the $50$ to the other side

$$2(x-5)^2 = 50.$$

Now, divide both side by two:

$$(x-5)^2 = 25.$$

Thus,

$$x - 5 = \pm 5 \qquad \text{and hence} \qquad x = 5 \pm 5 = 0, 10.$$

Thus $x = 0$ and $x = 10$ are the two roots.

**Algebraic Method #3 (quadratic equation):** Everyone should try to memorize this formula. The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.}$$

For example, if we want to find the roots of $x^2 - x - 6$, then we do

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2(1)} = \frac{1 \pm \sqrt{1 + 24}}{2} = \frac{1 \pm \sqrt{25}}{2} = \frac{1 \pm 5}{2} = \frac{6}{2}, \frac{-4}{2} = 3, -2.$$

**Tip!** If you have to solve many equations using the same quadratic function, for example you want to find out when $Q(x) = 0$, $Q(x) = 1$, $Q(x) = 2$, *etc.* you should use method 2. Once you put $Q(x)$ in standard form, it is easy to solve for each of the different values.

Of course you can also use your graphing calculator to find zeros