

How to Find Equations for Exponential Functions

William Cherry

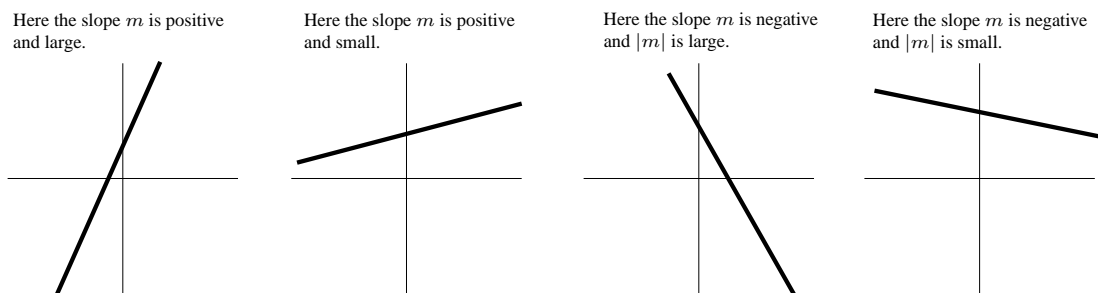
Introduction. After linear functions, the second most important class of functions are what are known as the “exponential” functions. Population growth, inflation, and radioactive decay are but a few examples of the various phenomenon that exponential functions can be used to model. Although exponential functions are very important, text books often do not go into as much detail about how to find equations for exponential functions as they do when they discuss linear functions. This tends to make students feel that exponential functions are much more complicated to work with than linear functions are. However, there are many similarities between the process used in finding the equation for a line and the process used to find an exponential equation. These notes are intended as a brief summary of the process used to find an exponential function. The hope is that by pointing out the similarities between the process used for finding equations of lines and the process used for finding equations of exponential functions, the student will find working with exponential functions less mysterious.

Base-intercept form of an exponential function. Recall that when working with equations for lines, it is often convenient to write the equation of the line in “slope-intercept” form – that is to write the equation in the form:

$$y = mx + b.$$

This is called “slope-intercept” form because the number m is the slope of the line and the number b is the y -intercept. The slope m determines if the graph is increasing or decreasing, and how quickly. A positive slope indicates the graph is increasing, and a negative slope indicates the graph is decreasing. A large positive slope indicates a graph that is increasing rapidly, whereas a small positive slope indicates a graph that increases slowly. The y -intercept b indicates where the graph crosses the y -axis, namely at the point $(0, b)$. Figure 1 illustrates the relationship between the graph of a linear function and the slope of the line.

Figure 1:

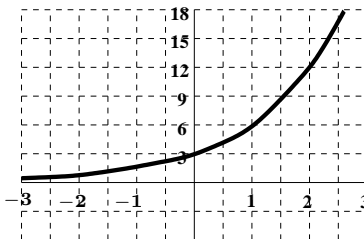


When working with exponential functions, it is often useful to work with the equation in “**base-intercept**” form:

$$y = P_0 a^x.$$

The function $y = 3 \cdot 2^x$, whose graph is shown in figure 2, is an example of an exponential function written in base-intercept form.

Figure 2:
Graph of $y = 3 \cdot 2^x$

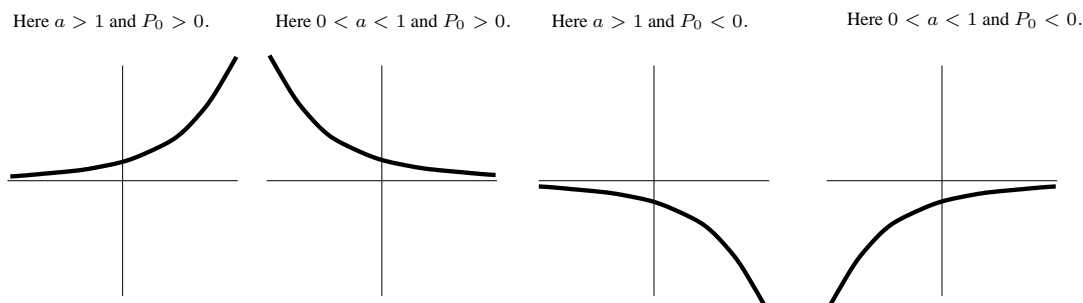


In base-intercept form, the number a is always a positive number, and is called the **base** of the exponential function. The number a is analogous to the slope of a line in that it determines if an exponential function is increasing,

decreasing, and how fast the function increases or decreases. The number P_0 is the y -intercept of the graph, meaning the graph crosses the y -axis at the point $(0, P_0)$.

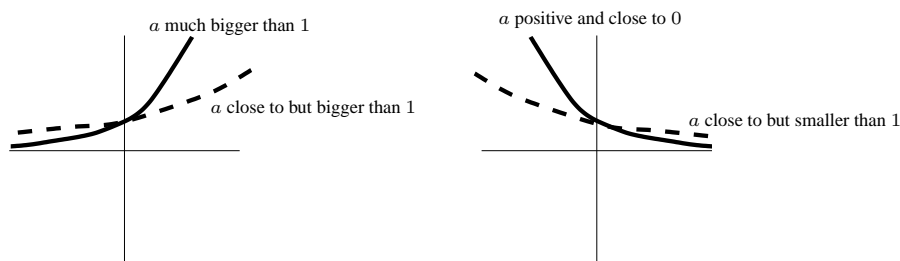
When $a > 1$, we call the function an **exponential growth** function. In this case, the graph moves away from the x -axis as x increases. That means the graph is increasing when $a > 1$ and $P_0 > 0$, and the graph is decreasing when $a > 1$ and $P_0 < 0$. When $0 < a < 1$, we call the function an **exponential decay** function. In this case, the graph moves toward the x -axis as x increases. That means the graph is decreasing when $a < 1$ and $P_0 > 0$, and the graph is increasing when $a < 1$ and $P_0 < 0$. This is illustrated in figure 3. The reader should focus on the case $P_0 > 0$,

Figure 3:



because this is the case that comes up most often in modeling problems. This is because the quantities that are usually modeled by exponential functions (population, money, radiation) tend to be positive. If a is very large, then the graph of $y = P_0 a^x$ moves away from the x -axis very quickly. If a is slightly bigger than one, then the graph moves away from the x -axis very slowly. If a is slightly smaller than one, then the graph approaches the x -axis very slowly. Finally, if a is close to zero, then the graph approaches the x -axis very quickly. This is illustrated in figure 4.

Figure 4:



How to find the equation of an exponential function passing through two points. Just as two points determine a line, there is only one exponential function that passes through any two given points. Recall how we find the equation of a line passing through the two points: (x_1, y_1) and (x_2, y_2) . We first find the slope m , which is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Let's look at this in a slightly different way. If the equation we are looking for is $y = mx + b$ and this equation is satisfied by the two points (x_1, y_1) and (x_2, y_2) , then we are looking for m and b that satisfy the system of equations

$$\begin{aligned} y_1 &= mx_1 + b \\ y_2 &= mx_2 + b. \end{aligned}$$

The idea is to get either m or b to cancel, and this is most easily done by subtracting the first equation from the second in order to get b to cancel:

$$y_2 - y_1 = mx_2 + b - (mx_1 + b) = mx_2 - mx_1 = m(x_2 - x_1).$$

Solving for m we find that

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

as expected. Once we have found the slope, we can find the y -intercept b just by plugging in one of the two points and solving for b .

With exponential functions we go through a very similar process. If we are looking for an equation of the form $y = P_0 a^x$ passing through the points (x_1, y_1) and (x_2, y_2) , then we are looking for a and P_0 satisfying the following system of equations:

$$\begin{aligned} y_1 &= P_0 a^{x_1} \\ y_2 &= P_0 a^{x_2}. \end{aligned}$$

Again, just as in the case of line, the idea is to do something so that either P_0 or a is canceled. In the line case we were able to cancel the b by subtracting the first equation from the second. This time we will cancel the P_0 by *dividing* the second equation by the first equation. Namely,

$$\frac{y_2}{y_1} = \frac{P_0 a^{x_2}}{P_0 a^{x_1}} = \frac{a^{x_2}}{a^{x_1}} = a^{x_2 - x_1}.$$

So, we have the equation

$$\frac{y_2}{y_1} = a^{x_2 - x_1}.$$

To solve an equation like this, we raise both sides to the $1/(x_2 - x_1)$ power to get

$$\left(\frac{y_2}{y_1}\right)^{\frac{1}{x_2 - x_1}} = (a^{x_2 - x_1})^{\frac{1}{x_2 - x_1}} = a^{\frac{x_2 - x_1}{x_2 - x_1}} = a^1 = a.$$

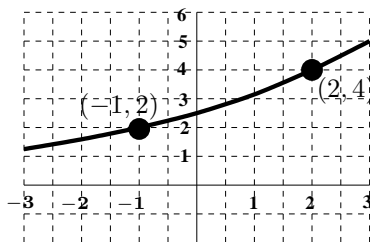
Thus, we are left with the equation

$$a = \left(\frac{y_2}{y_1}\right)^{\frac{1}{x_2 - x_1}},$$

which is worth memorizing. After we find the base a , we do just as we do in the linear case and plug in one of our two points and solve for the y -intercept P_0 .

Example. Find an equation for the exponential function passing through the points: $(-1, 2)$ and $(2, 4)$, as shown in figure 5.

Figure 5:



Solution. It's always helpful to think about what we would do in the case of a line. In the line case, we always find the slope first. Since for exponential functions, the base is analogous to the slope, we should find the base a first. To do that we use the formula

$$a = \left(\frac{y_2}{y_1}\right)^{\frac{1}{x_2 - x_1}}$$

that we discussed above. Plugging in our two points $(-1, 2)$ and $(2, 4)$ to this formula, we get

$$a = \left(\frac{4}{2}\right)^{\frac{1}{2 - (-1)}} = 2^{\frac{1}{3}} = \sqrt[3]{2} \approx 1.25992105.$$

Now, to solve for P_0 we plug one of the two points into the equation:

$$y = P_0 (\sqrt[3]{2})^x.$$

Using the point (2, 4) for example, we get

$$4 = P_0(\sqrt[3]{2})^2$$

so

$$P_0 = \frac{4}{(\sqrt[3]{2})^2} \approx 2.5198421.$$

Thus, the desired equation is:

$$y = \frac{4}{(\sqrt[3]{2})^2} (\sqrt[3]{2})^x \approx 2.5198421(1.25992105)^x.$$

Note that when working with decimals, it is important to use several more decimal places in the base a that you desire to be correct in your final answer because any round-off error can be greatly magnified by raising to the x -th power.

Point-base form for an exponential function. You may recall when working with lines that sometimes working with the “point-slope” form of the equation can be more convenient than working with the “slope-intercept” form. Recall that “point-slope” form is:

$$y = y_1 + m(x - x_1).$$

Recall that this is called “point-slope form” because you can see the slope m and the point (x_1, y_1) in the equation.

In the case of exponential functions it is sometimes convenient to write the exponential function in “point-base” form. Point base form is:

$$y = y_1 a^{x-x_1},$$

and is called “point-base” form because you can see the base a and the point (x_1, y_1) in the equation.

To find the point-base form of an exponential equation, you first find the base using the same formula:

$$a = \left(\frac{y_2}{y_1} \right)^{\frac{1}{x_2 - x_1}},$$

and then you plug in one of the points.

Example. Find an equation for the exponential function passing through the points: (1, 4) and (3, 12).

Solution. As before, we begin by finding the base:

$$a = \left(\frac{12}{4} \right)^{\frac{1}{3-1}} = \sqrt{3}.$$

Then, plugging in the point (1, 4) to the “point-base” form of an exponential equation we get:

$$y = 4(\sqrt{3})^{x-1}.$$

This can also be written as

$$y = 4 \cdot 3^{(x-1)/2},$$

which some people feel looks neater.

How to tell if a table of data can be modeled by a linear function, by an exponential function, or by neither. Consider the following tables:

x	$y = A(x)$
1	2.00
2	5.00
3	12.50
4	31.25

x	$y = B(x)$
2	1.6
4	1.8
6	2.0
8	2.2

x	$y = C(x)$
3	2.25
6	9.00
9	20.25
12	36.00

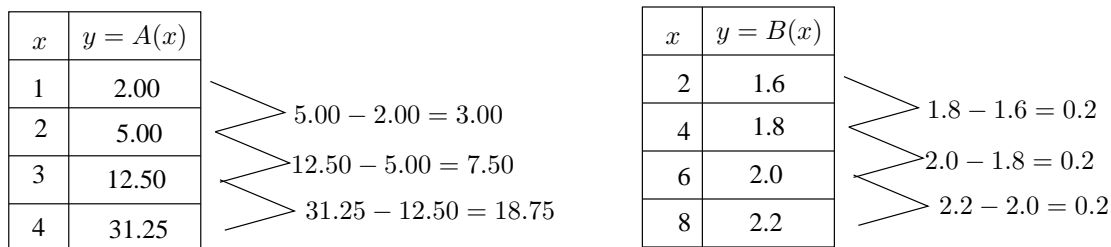
x	$y = D(x)$
1	10
5	7
7	4
9	1

x	$y = E(x)$
3	16
5	8
7	4
9	2

Each of these tables has an important feature: the x -values are “equally spaced.” In other words, the x values always increase by a constant amount: in the first table the x values increase by 1, in the second table by 2, in the third table by 3, in the fourth table by 2, and in the last table by 2. When given a table of data with equally spaced x -values, it is easy to tell if the data should be modeled with a linear function, with an exponential function, or with neither type of function.

First, we will consider the case of linear functions. If we have a linear function, then because the slope is rise over run, equally spaced increases in the x -variable would result in equally spaced increases in the y -variable. Thus, to see if a table of data with equally spaced x -values represents a linear function, we just have to check if the y -values are also equally spaced. In the first table, the y -values are not equally spaced: $5 - 2 = 3 \neq 7.5 = 12.5 - 5$. In the second table, note that the y -values are equally spaced. The difference between any two successive y -values is equal to 0.2. This is illustrated in figure 6. This indicates that $B(x)$ is a linear function, and moreover its slope is change in y over change in x , which is $0.2/2$ or 0.1 . The third table clearly does not have equally spaced y -values, so $C(x)$ is not a linear function. The y -values in the fourth table are equally spaced, decreasing by 3 each time, so $D(x)$ is a linear function with slope $3/2$. And, the y -values in the last table are clearly not equally spaced.

Figure 6:



If a function were exponential, then using the formula

$$a = \left(\frac{y_2}{y_1} \right)^{\frac{1}{x_2 - x_1}},$$

we would have to get the same value for a no matter which two points from the table we chose. When all our x -values are equally spaced, as in the tables above, then $x_2 - x_1$ is always the same for any two successive points in the table, so we will always get the same a if y_2/y_1 is always the same for any two successive y -values. Thus, to detect exponential functions given a table of data with equally spaced x -values, we look for a *common ratio* among the successive y -values instead of a common difference. Since we already know $B(x)$ and $D(x)$ are linear functions, we know they are not exponential. In the case of the first table, we see that

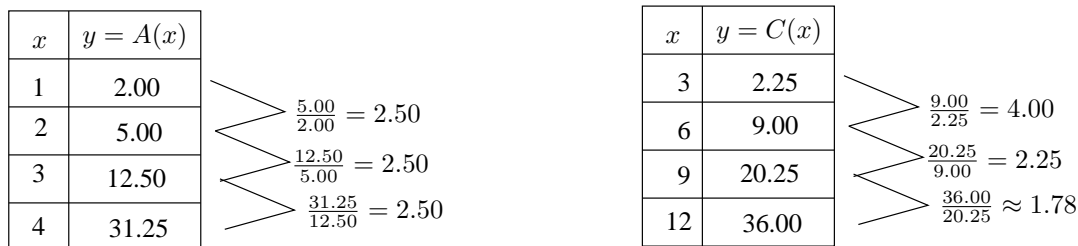
$$\frac{5.00}{2.00} = \frac{12.50}{5.00} = \frac{31.25}{12.50} = 2.50.$$

Thus, $A(x)$ is an exponential function with base 2.5. If we examine the successive ratios of the y -values in table C, we see

$$\frac{9.00}{2.25} \neq \frac{20.25}{9.00} \neq \frac{36.00}{20.25},$$

and so $C(x)$ is neither a linear function nor an exponential function. This is illustrated in figure 7.

Figure 7:



Finally, we see that the y -values in the last table have a common ratio of $\frac{1}{2}$, and so $E(x)$ is an exponential function with base

$$a = \left(\frac{1}{2} \right)^{\frac{1}{2}} = \sqrt{\frac{1}{2}}.$$

Note we take the square root because the difference in the x -values is two.

What to do if the x -values are not equally spaced. Now consider the following table of data.

x	1985	1990	1994	1996	1998
$f(x)$	24.0	18.0	13.2	10.8	8.4
$g(x)$	24.000	38.652	56.591	68.475	82.855
$h(x)$	24	23	22	21	20

In this case, it is not as easy to tell if the functions are exponential or linear because the x -values are not equally spaced.

In this case to check that a function is linear we have to compute the slope with each successive pair of points and see if we get the same answer each time. For $f(x)$ we want to compare the following values

$$\frac{18.0 - 24.0}{1990 - 1985} = -1.2, \quad \frac{13.2 - 18.0}{1994 - 1990} = -1.2, \quad \frac{10.8 - 13.2}{1996 - 1994} = -1.2, \quad \text{and} \quad \frac{8.4 - 10.8}{1998 - 1996} = -1.2.$$

Since we always get the same value of -1.2 , we know that the graph of $f(x)$ is a line with slope -1.2 . To find an equation for $f(x)$, we simply plug in any one of the points, say $(1985, 24.0)$ to get

$$f(x) = 24.0 - 1.2(x - 1985).$$

For $g(x)$, we compare

$$\frac{38.652 - 24.000}{1990 - 1985} = 2.930, \quad \frac{56.591 - 38.652}{1994 - 1990} = 4.485, \quad \frac{68.475 - 56.591}{1996 - 1994} = 5.942, \quad \text{and} \quad \frac{82.855 - 68.475}{1998 - 1996} = 7.19.$$

These are clearly not equal, so $g(x)$ is not linear. Finally, for $h(x)$, we compare

$$\frac{23 - 24}{1990 - 1985} = -0.2, \quad \frac{22 - 23}{1994 - 1990} = -0.25, \quad \frac{21 - 22}{1996 - 1994} = -0.5, \quad \text{and} \quad \frac{20 - 21}{1998 - 1996} = -0.5.$$

Again, these are not all the same, and so $h(x)$ is not a linear function.

To check to see if any of these functions are exponential, we compute the base using the formula

$$a = \left(\frac{y_2}{y_1} \right)^{\frac{1}{x_2 - x_1}}$$

for each successive pair of points in the table, and see if always get the same answer. We'll skip $f(x)$ since we already know $f(x)$ is linear and not exponential. For $g(x)$, we compare

$$\left(\frac{38.652}{24.000} \right)^{\frac{1}{1990 - 1985}} = 1.100, \quad \left(\frac{56.591}{38.652} \right)^{\frac{1}{1994 - 1990}} = 1.100, \quad \left(\frac{68.475}{56.591} \right)^{\frac{1}{1996 - 1994}} = 1.100, \quad \text{and} \quad \left(\frac{82.855}{68.475} \right)^{\frac{1}{1998 - 1996}} = 1.100.$$

Since all these numbers are the same, we know $g(x)$ is an exponential function with base $a = 1.1$. To find an equation for $g(x)$, we plug in any one of the points in the table, say $(1985, 24.000)$, and we get

$$g(x) = 24(1.1)^{x - 1985}.$$

Finally, for $h(x)$ we compare the numbers

$$\left(\frac{23}{24} \right)^{\frac{1}{1990 - 1985}} = 0.9915, \quad \left(\frac{22}{23} \right)^{\frac{1}{1994 - 1990}} = 0.9889, \quad \left(\frac{21}{22} \right)^{\frac{1}{1996 - 1994}} = 0.9770, \quad \text{and} \quad \left(\frac{20}{21} \right)^{\frac{1}{1998 - 1996}} = 0.9759.$$

Since these numbers are not the same, we see that $h(x)$ is not an exponential function.

An example of a word problem using an exponential function.

Example. A biologist starts a bacteria culture growing. The culture began with 1000 cells. Three hours later the biologist returned to find 5000 cells in the culture. Assuming the bacteria culture grows exponentially, how many cells will there be in the culture 10 hours after the culture began growing?

Solution. Let t be the number of hours since the culture began to grow, and let $B(t)$ denote the amount of bacteria present at time t . We are told that $B(0) = 1000$ and $B(3) = 5000$. We are also told to assume that $B(t)$

grows exponentially, so that we are looking for an equation of the form $B(t) = B_0 a^t$. Since we know two points on the graph, we can determine this equation. We start by finding a , using the formula

$$a = \left(\frac{5000}{1000} \right)^{\frac{1}{3-0}} = 5^{1/3}.$$

Thus, our equation is of the form

$$B(t) = B_0(5^{1/3})^t = B_0(5)^{t/3}.$$

In this case it is easy to find the y -intercept B_0 since we already know that there were 1000 bacteria cells to start with. Thus $B_0 = 1000$, and so our equation is

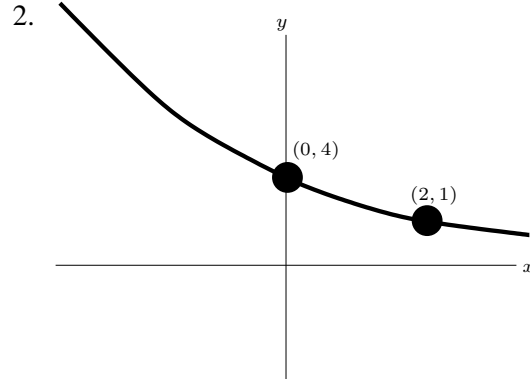
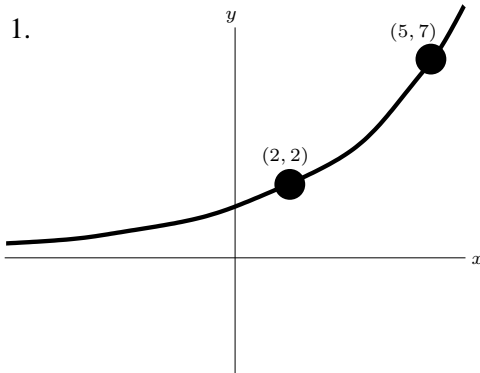
$$B(t) = 1000(5)^{t/3}.$$

The question asks us how many bacteria cells will be present when $t = 10$, so we just plug in 10 for t to get

$$B(10) = 1000(5)^{10/3} \approx 213,747 \text{ cells.}$$

Exercises

In problems 1 and 2, find equations for the graphs shown.



In each of problems 3 and 4, there are three functions defined by a table. In each case one of the functions is linear, one of the functions is exponential, and one of the functions is neither. In each case, (a) identify the linear and exponential functions, (b) find a formula for the linear function, (c) find a formula for the exponential function, and (d) try to guess a formula for the third function.

3.

x	$f(x)$	$g(x)$	$h(x)$
-2	4	0	1/9
0	0	1	1/3
2	4	2	1
4	16	3	3
6	36	4	9

4.

x	$f(x)$	$g(x)$	$h(x)$
0	100.00	36.25	0
3	90.00	34.20	27
6	81.00	32.15	216
9	72.90	30.10	729
12	65.61	28.05	1728

5. Assume that the median price P of a home rose from \$50,000 in 1970 to \$100,000 in 1990. Let t be the number of years since 1970.

- Assume the increase in housing prices has been linear. Give an equation for a line representing the price P in terms of t . Use this equation to complete column (a) of the table below. Work with price in units of \$1000.
- If instead, the housing prices have been rising exponentially, determine an equation of the form $P = P_0 a^t$ which would represent the price of a house as a function of t . Again, work with price in units of \$1000 and use your formula to complete column (b) of the table below.
- On the same set of axes, sketch both functions in parts (a) and (b).

t	(a)	(b)
0	50	50
10		
20	100	100
30		
40		

Selected Answers: (1) $y = 2(7/2)^{x/2} \approx 0.8675968(1.5182945)^x$ (3) $f(x) = x^2$, $g(x) = 1 + 2/x$, $h(x) = (3)^{x/2}$.

Relative-rate-intercept form of an exponential function. Previously, we discussed “base-intercept” form of an exponential equation. Now, we will discuss another standard form for exponential equations. The number $e \approx 2.71828$ is an important number, and as we have seen with continuously compounded interest, it is a natural base to use when working with continuous rates, also called “relative rates.” Any exponential function can be written using the natural base e . When this is done, the equation is said to be in “**relative-rate-intercept**” or “**continuous-rate-intercept**” form:

$$y = P_0 e^{rx}.$$

In relative-rate-intercept form, the number r is called the **relative rate** (or **continuous rate**) of the exponential function. It is analogous to the slope of a line in that it determines if an exponential function is increasing or decreasing, and how fast the function increases or decreases. The number P_0 is again the y -intercept of the graph, meaning the graph crosses the y -axis at the point $(0, P_0)$.

If $r > 0$, then $y = P_0 e^{rx}$ is an exponential growth function, and if $r < 0$, then $y = P_0 e^{rx}$ is an exponential decay function.

How to convert between base-intercept-form and relative-rate-intercept form. Suppose we have an exponential equation in base-intercept form:

$$y = P_0 a^x.$$

How can we convert this to relative-rate-intercept form:

$$y = P_0 e^{rx}?$$

Well, since we want the two functions to be equal, we set

$$P_0 a^x = P_0 e^{rx}.$$

Now, note that the P_0 's cancel, and we are left with

$$a^x = e^{rx} = (e^r)^x.$$

Thus,

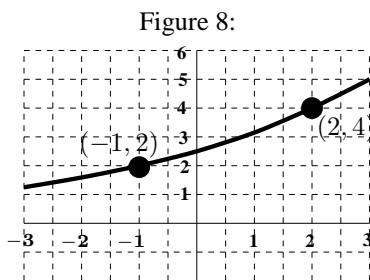
$$a = e^r \quad \text{and} \quad r = \ln a.$$

For example, if we want to write $y = 3 \cdot 2^x$ in relative-rate-intercept form, then we use the formula $r = \ln a$ with $a = 2$ to conclude that

$$y = 3e^{(\ln 2)x} \approx 3e^{0.69315x}.$$

How to find the equation of an exponential function passing through two points in relative-rate-intercept form.

Example. Find an equation for the exponential function passing through the points: $(-1, 2)$ and $(2, 4)$, as shown in figure 8.



Solution. Recall from before, that the first thing we wanted to do was find the base. To do that we used the formula

$$a = \left(\frac{y_2}{y_1} \right)^{\frac{1}{x_2 - x_1}}.$$

This time, we use this formula together with the formula

$$r = \ln a$$

to get

$$r = \ln a = \ln \left(\frac{y_2}{y_1} \right)^{\frac{1}{x_2 - x_1}} = \frac{\ln y_2 - \ln y_1}{x_2 - x_1}.$$

Thus, we are left with the formula

$$r = \frac{\ln y_2 - \ln y_1}{x_2 - x_1}.$$

Plugging in the points $(-1, 2)$ and $(2, 4)$ to this equation gives us

$$r = \frac{\ln 4 - \ln 2}{2 - (-1)} \approx \frac{0.693147}{3} = 0.23105.$$

Thus, our equation is of the form

$$y = P_0 e^{0.23105x}.$$

To find P_0 , as always, we plug in one of the two points. For example, plugging in the point $(2, 4)$, we find that

$$4 = P_0 e^{0.23105 \cdot 2},$$

so

$$P_0 = \frac{4}{e^{0.23105 \cdot 2}} = 2.5198,$$

and our final equation is

$$y = 2.5198e^{0.23105x}.$$

An example of a word problem using an exponential function. Let's look back at the word problem we did a few pages back, but this time using continuous-rate-intercept form for our exponential equations.

Example. *A biologist starts a bacteria culture growing. The culture began with 1000 cells. Three hours later the biologist returned to find 5000 cells in the culture. Assuming the bacteria culture grows exponentially, how many cells will there be in the culture 10 hours after the culture began growing?*

Solution. Let t be the number of hours since the culture began to grow, and let $B(t)$ denote the amount of bacteria present at time t . We are told that $B(0) = 1000$ and $B(3) = 5000$. We are also told to assume that $B(t)$ grows exponentially, so that we are looking for an equation of the form $B(t) = B_0 e^{rt}$. Since we know two points on the graph, we can determine this equation. We start by finding the relative (or continuous) hourly growth rate r , using the formula

$$r = \frac{\ln 5000 - \ln 1000}{3 - 0} = 0.5364793.$$

Thus, our equation is of the form

$$B(t) = B_0 e^{0.5364793t}.$$

In this case it is easy to find the y -intercept B_0 since we already know that there were 1000 bacteria cells to start with. Thus $B_0 = 1000$, and so our equation is

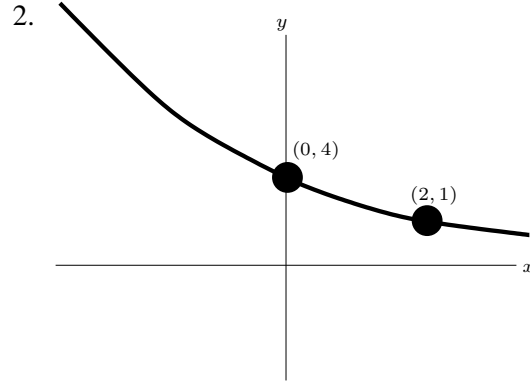
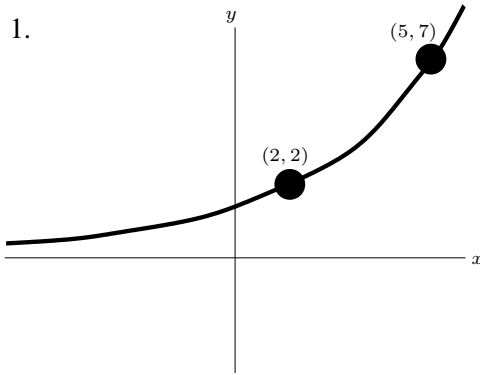
$$B(t) = 1000e^{0.5364793t}.$$

The question asks us how many bacteria cells will be present when $t = 10$, so we just plug in 10 for t to get

$$B(10) = 1000e^{0.5364793 \cdot 10} \approx 213,747 \text{ cells.}$$

Exercises

In problems 1 and 2, find equations for the graphs shown. Express your answers in **relative rate-intercept form**.



3. If the annual rate of inflation is 3.5%, what is the relative rate of inflation?
4. If the relative rate of inflation is 2% per year, what is the annual rate of inflation?
5. If the annual rate of inflation is 3%, by what percentage will prices increase over a 10 year period? What if the relative rate of inflation were 3% per year?
6. If the annual rate of inflation is 3%, how long will it take for prices to double? What about if the relative rate of inflation were 3% per year?

Selected Answers:
 (1) $y \approx 0.8675968e^{0.41758766x}$
 (3) $\ln(1.03) \approx 2.95588\%$
 (5) $(1.03)_{10} \approx 1.34391638$ – an increase of 34.391638%.

$e^{0.03 \cdot 10} \approx 1.3498588$ – an increase of 34.98588%.