These notes outline how to prove the assertion in part (a) on the tube problem on the second test that given a smooth simple closed curve $C$ in space, for $r$ small enough, the tube of radius $r$ about $C$ is a regular surface. As in the test problem, we will assume $C$ has everywhere positive curvature.

Checking that $\vec{x}$ is differentiable and that if $r$ is small enough, then $d\vec{x}$ has rank two is straightforward. If we can show that for $r$ sufficiently small that $\vec{x}$ is injective, then we can conclude the surface is a regular surface by using Proposition 2 on page 79 of Do Carmo.

The fact that $d\vec{x}$ has rank two implies that $\vec{x}$ is locally injective, meaning that given $(s_0, \theta_0)$, $\vec{x}(s, \theta)$ is injective if we keep $s$ near $s_0$ and $\theta$ near $\theta_0$. This is part of the content of Proposition 2 on page 79 and is a consequence of the Implicit Function Theorem. However, it is not clear how this alone can be used. In the special case of the tube, we can work more explicitly and we can find that $\vec{x}$ is locally injective if we do not change $s$ very much from $s_0$ but we let $\theta$ be arbitrary, and we can explicitly estimate how far we can move $s$ away from $s_0$.

Claim. Let $k = \max\{k(s) : s \in [0, L]\} < \infty$. Let $r_0 < 1/k$. Let $\varepsilon < 1/k - r_0$ and let $r < r_0$. Then, if $|s - s_0| < \varepsilon$ and $\vec{x}(s, \theta) = \vec{x}(s_0, \theta_0)$, then $s = s_0$ and $\theta = \theta_0$. Note that it is important that $\varepsilon$ does not depend on $r$.

Proof. Stephen, in his test solution, suggested this could be proven in a manner similar to the proof of Lemma 3 on the first test. Indeed, let $p$ denote the point $\vec{x}(s_0, \theta_0)$. Consider the function

$$ f(s) = \langle \alpha(s) - p, \alpha(s) - p \rangle $$

and note that $f(s_0) = r^2$ because the point $p$ is distance $r$ from the point $\alpha(s_0)$ by the definition of $\vec{x}$. Also,

$$ f'(s_0) = 2 < \alpha'(s_0), \alpha(s_0) - p > $$

and note that $\alpha'(s_0)$ is orthogonal to $\alpha(s_0) - p$ again by the definition of $\vec{x}$. Hence $s_0$ is a critical point for $f$. Differentiating again, we see

$$ f''(s) = 2[< \alpha'(s), \alpha'(s) > + < \alpha''(s), \alpha(s) - p >]. $$

Because $s$ is an arc-length parameter,

$$ < \alpha'(s), \alpha'(s) >= 1. $$

Also, by Cauchy-Schwarz (or because $|\cos| \leq 1$),

$$ |< \alpha''(s), \alpha(s) - p >| \leq |\alpha''(s)| \cdot |\alpha(s) - p| \leq k|\alpha(s) - p|. $$
Now, by the triangle inequality,

$$|\alpha(s) - p| = |\alpha(s) - \alpha(s_0) + \alpha(s_0) - p| \leq |\alpha(s) - \alpha(s_0)| + |\alpha(s_0) - p|.$$ 

We know $|\alpha(s_0) - p| = r < r_0$ and because $s$ is an arc-length parameter, we know

$$|\alpha(s) - \alpha(s_0)| \leq |s - s_0| < \varepsilon.$$

Hence,

$$f''(s) > 2(1 - k(r_0 + \varepsilon)) > 0,$$

by our choice of $\varepsilon$. Thus, $s_0$ is the unique local minimum for $f(s)$ in the interval $(s_0 - \varepsilon, s_0 + \varepsilon)$ and so $f(s) > f(s_0) = r^2$, and so $\bar{x}(s, \theta)$ can equal $\bar{x}(s_0, \theta_0)$ only if $|s - s_0| \geq \varepsilon$ or $s = s_0$. If $s = s_0$, it is easy to see we must have $\theta = \theta_0$.

The above claim is the main thing to prove. It tells us that $\bar{x}$ is locally injective, with an explicit determination of locally. Now we just need to see that if $r$ is small, we cannot have $\bar{x}$ be non-injective for $s$ far away from $s_0$. To do this, we break $C$ up into finitely many pieces so that we know $\bar{x}$ is injective on each piece and then make $r$ smaller than half the distance between pieces. Explicitly, choose $s_1, \ldots, s_n$ so that

$$[0, L] \subseteq \bigcup_{j=1}^{n} (s_j - \varepsilon/2, s_j + \varepsilon/2)$$

and let

$$d_j = \min\{|\alpha(s) - \alpha(t)| : s \in [s_j - \varepsilon/2, s_j + \varepsilon/2] \text{ and } t \notin (s_j - \varepsilon, s_j + \varepsilon)\} > 0.$$

Then if $2r < \min d_j$, we must have $\bar{x}$ injective.